D-MATH Functional analysis II Spring 2019
Prof. Dr. Manfred Einsiedler
Andreas Wieser

Solutions for exercise sheet 10

1. Let $\mathcal{A}$ be as in the hint and let $\chi \in \sigma(\mathcal{A})$. If $a' \in \mathcal{A}'$ is any element and $n \in \mathbb{N}$ is such that $(a')^n = 0$ then

$$0 = \chi((a')^n) = \chi(a')^n$$

and so $\chi(a') = 0$. This shows that χ vanishes identically on $\mathcal{A}'$. Thus, χ agrees with the character

$$\mathcal{A} = \mathbb{C} \times \mathcal{A}' \ni (\alpha, a) \mapsto \alpha$$

by $\mathbb{C}$-linearity. This shows that $\sigma(\mathcal{A})$ consists of a point and is thus non-empty and compact.

2. **a)** By Proposition 11.38 the Pontryagin dual $\widehat{\mathbb{Z}}$ can be identified with the Gelfand dual $\sigma(\ell^1(\mathbb{Z}^d))$ under the map

$$\chi \in \widehat{\mathbb{Z}^d} \mapsto \Big(a \in \ell^1(\mathbb{Z}^d) \mapsto \int_{\mathbb{Z}^d} a\chi \,\mathrm{d}m_{\mathbb{Z}^d} = \sum_{n \in \mathbb{Z}^d} a_n \chi(n) \Big)$$

where we used that $\mathbb{Z}^d$ is discrete so that the counting measure is its Haar measure. Combining this with the identification $\mathbb{T}^d \simeq \widehat{\mathbb{Z}^d}$ we obtain that the Gelfand dual is identified with the torus under the map

$$t \in \mathbb{T}^d \mapsto \Big(a \in \ell^1(\mathbb{Z}^d) \mapsto \sum_{n \in \mathbb{Z}^d} a_n \chi_t(n) = \sum_{n \in \mathbb{Z}^d} a_n \mathrm{e}^{2\pi \mathrm{i} n \cdot t} \Big).$$

Note that for $t \in \mathbb{T}^d$ the latter character exactly evaluates the Fourier series that a defines at the value t.

Also by Proposition 11.38 the Gelfand transform

$$\ell^1(\mathbb{Z}^d) \to C(\sigma(\ell^1(\mathbb{Z}^d)), \quad a \mapsto a^\circ$$

corresponds to the Fourier back transform

$$\ell^1(\mathbb{Z}^d) \to C(\widehat{\mathbb{Z}}^d), \quad a \mapsto \check{a}.$$

under the above identification of the Gelfand dual with the Pontryagin dual. Notice that for $t \in \mathbb{T}^d$ and $a \in \ell^1(\mathbb{Z}^d)$

$$\check{a}(\chi_t) = \int_{\mathbb{Z}^d} a\chi_t \,\mathrm{d}m_{\mathbb{Z}^d} = \sum_{n \in \mathbb{Z}^d} a_n \mathrm{e}^{2\pi \mathrm{i} n \cdot t}.$$

We put everything together and obtain that the map $\ell^1(\mathbb{Z}^d) \to C(\mathbb{T}^d)$ obtained under the above identifications is given at $f \in \ell^1(\mathbb{Z}^d)$ by

$$t \in \mathbb{T}^d \mapsto \check{f}(\chi_t) = \sum_{n \in \mathbb{Z}^d} f(n) \mathrm{e}^{2\pi \mathrm{i} n \cdot t}$$

as claimed.

b) Recall that the spectrum $\sigma(a)$ of $a \in \ell^1(\mathbb{Z}^d)$ is the set of values $\chi(a)$ for $\chi \in \sigma(\ell^1(\mathbb{Z}^d))$. Using the Gelfand dual

$$\begin{aligned}\sigma(a) &= \{\chi(a) : \chi \in \sigma(\ell^1(\mathbb{Z}^d))\} = \{a^\circ(\chi) : \chi \in \sigma(\ell^1(\mathbb{Z}^d))\} \\ &= a^\circ\Big(\sigma(\ell^1(\mathbb{Z}^d))\Big).\end{aligned}$$

Part a) identifies the map a° with the map

$$\tilde{a} : t \in \mathbb{T}^d \mapsto \sum_{n \in \mathbb{Z}^d} a_n \mathrm{e}^{2\pi \mathrm{i} n \cdot t}.$$

Thus,

$$\sigma(a) = a^\circ\Big(\sigma(\ell^1(\mathbb{Z}^d))\Big) = \tilde{a}(\mathbb{T}^d).$$

c) If $f \in C(\mathbb{T}^d)$ has an absolutely convergent Fourier series and $a \in \ell^1(\mathbb{Z}^d)$ is the sequence of Fourier coefficients, then $f = \tilde{a}$ and by b) $\sigma(a) = f(\mathbb{T}^d)$. The assumption that f does not attain zero thus means that $0 \notin \sigma(a)$. In other words, a is invertible. Let $b \in \ell^1(\mathbb{Z}^d)$ be its inverse. Then $f\tilde{b} = \tilde{a}\tilde{b} = \widetilde{ab} = \tilde{\delta}_0 = 1$ and $\frac{1}{f}$ is equal to $\tilde{b}$. The latter however has an absolutely convergent Fourier series by definition.

3. **a)** It is straightforward to check that χ_t is a character for any $t \in \mathbb{T}^d$. If $\chi_{t_1} = \chi_{t_2}$ then $t = t_1 - t_2$ satisfies that $\chi_t = 1$. This means that $x \cdot t \in \mathbb{Z}^d$ for every $x \in \mathbb{R}^d$. Since $\mathbb{R}^d$ is connected, $\mathbb{Z}^d$ is discrete, and the map $x \mapsto x \cdot t$ is continuous, $x \cdot t = 0$ for every $x \in \mathbb{R}^d$. Thus, $t = 0$ and $t_1 = t_2$ which proves the desired injectivity.

b) Let us first assume that $d = 1$. We use the hint and note that

$$\int_s^{s+\delta} \chi(x)\,\mathrm{d}x = \int_0^{s+\delta} \chi(x)\,\mathrm{d}x - \int_0^{s} \chi(x)\,\mathrm{d}x$$

is a continuously differentiable function in s by standard calculus. On the other hand, we may choose $\delta > 0$ small enough so that $\chi|_{[0,\delta]}$ is a positive function. Indeed, χ is continuous and $\chi(0) = 1$. In particular, $\int_0^\delta \chi(x)\,\mathrm{d}x$ is non-zero. Therefore,

$$\chi(s) = \frac{\int_s^{s+\delta} \chi(x)\,\mathrm{d}x}{\int_0^\delta \chi(x)\,\mathrm{d}x}$$

is continuously differentiable as a function of s.

Since χ is a character, we have $\chi(s+t) = \chi(s)\chi(t)$ for every $s, t \in \mathbb{R}^d$. Taking the derivative with respect to s we get

$$\chi'(s+t) = \chi'(s)\chi(t)$$

for every $s, t \in \mathbb{R}^d$ where χ' is the derivative of χ. In particular, we get

$$\chi'(t) = \chi'(0)\chi(t).$$

The unique solution of this ordinary differential equation is given by

$$t \mapsto \chi(t) = \mathrm{e}^{\chi'(0)t}.$$

In particular, χ is smooth. Also, since $|\chi(t)| = 1$ we get that

$$1 = |\chi(1)| = \left|\mathrm{e}^{\chi'(0)}\right| = \mathrm{e}^{\mathrm{Re}(\chi'(0))}$$

and so $\chi'(0)$ is purely imaginary. This shows that χ has the desired form and smoothness and proves part b) for $d = 1$.

For $d > 1$ we note that

$$\chi(t) = \chi(t_1 e_1 + \ldots + t_d e_d) = \chi(t_1 e_1) \cdots \chi(t_d e_d)$$

By the claim for $d = 1$ the right hand side is smooth and thus χ is smooth. Also, there exists for every $i \in \{1, \ldots, d\}$ some $x_i \in \mathbb{R}^d$ such that $\chi(t_i e_i) = \mathrm{e}^{2\pi \mathrm{i} x_i t_i}$ for all $t \in \mathbb{R}^d$. Thus,

$$\chi(t) = \mathrm{e}^{2\pi \mathrm{i} x_1 t_1} \cdots \mathrm{e}^{2\pi \mathrm{i} x_d t_d} = \mathrm{e}^{2\pi \mathrm{i} x \cdot t}.$$

c) Let $\chi_t \in \widehat{\mathbb{R}^d}$, $\varepsilon > 0$ and $K \subset \mathbb{R}^d$ compact be given. We need to find $\delta > 0$ such that for any $t' \in \mathbb{R}^d$ with $\|s - t\| < \delta$ we have

$$\chi_s \in U_{K,\varepsilon}(\chi_t) = \{\chi \in \widehat{\mathbb{R}^d} : \|\chi - \chi_t\|_{K,\infty} < \varepsilon\}$$

For this, we use continuity of $\alpha \in \mathbb{R} \mapsto \mathrm{e}^{2\pi \mathrm{i}\alpha}$ at zero and the Cauchy-Schwarz inequality. Let $\delta_1 > 0$ be such that $|\mathrm{e}^{2\pi\mathrm{i}\alpha} - 1| < \varepsilon$ for all α with $|\alpha| < \delta_1$. Furthermore, note that for any $x \in K$

$$|x \cdot (t - s)| \leq \|x\|\|t - s\| \ll \|t - s\|$$

by compactness. We choose $\delta > 0$ such that $|x \cdot (t - s)| < \delta_1$ for all $x \in K$ and $s \in \mathbb{R}^d$ with $\|s - t\| < \delta$. Together, these two estimate show that for any $s \in \mathbb{R}^d$ with $\|s - t\| < \delta$

$$\begin{aligned} \|\chi_t - \chi_s\|_{K,\infty} &= \sup_{x \in K} |\chi_t(x) - \chi_s(x)| = \sup_{x \in K} |\chi_{t-s}(x) - 1| \\ &= \sup_{x \in K} \left|\mathrm{e}^{2\pi\mathrm{i}x\cdot(t-s)} \leq \varepsilon \right. \end{aligned}$$

as we needed to show.

4. **a)** Since the discrete spectrum is empty, we have that the kernel of $T - \lambda I$ is trivial for any λ and so the approximate point spectrum is equal to the approximate spectrum. We will compute the former.

Suppose first that $\lambda \in \mathbb{C}$ has absolute value one. We need to find for any $\varepsilon > 0$ some $v \in \ell^2(\mathbb{N})$ with norm one and with $\|(T - \lambda I)v\| < \varepsilon$. Note that

$$(T - \lambda I)v = (-\lambda v_1,\ v_1 - \lambda v_2, \ldots).$$

Let us consider for some $N \in \mathbb{N}$ to be chosen later the vector $v \in \ell^2(\mathbb{N})$ given by $v_1 = \varepsilon$, $v_2 = \frac{v_1}{\lambda}$, $v_3 = \frac{v_2}{\lambda}$ and so forth up to $v_N, v_{N+1}, \ldots$ which we all set equal to zero. Then $((T - \lambda I)v)_n = 0$ whenever $n \neq 1, N$, $((T - \lambda I)v)_1 = -\lambda\varepsilon$ and

$$((T - \lambda I)v)_N = v_{N-1} = \frac{v_{N-2}}{\lambda} = \ldots = \lambda^{-(N-2)} v_1.$$

This shows that

$$\|(T - \lambda I)v\|^2 = | - \lambda v_1|^2 + |\lambda^{-(N-2)} v_1| = 2|v_1|^2 = 2\varepsilon^2.$$

Furthermore,

$$\|v\|^2 = |v_1|^2 + \ldots + |v_{N-1}|^2 = (N - 1)\varepsilon^2.$$

Therefore, choosing N of order $\frac{1}{\sqrt{\varepsilon}}$ gives the desired estimates. Since $\varepsilon > 0$ was arbitrary we conclude.

Suppose that now that there is $\lambda \in \sigma_{\text{approx}}(T) \setminus \mathbb{S}^1$. We note that $\sigma_{\text{approx}}(T) \subset \sigma(T) \subset \overline{B_1(0)}$ and so $|\lambda| < 1$. Note that for any sequence $v^{(k)}$ of elements of $\ell^2(\mathbb{N})$ of norm one we have

$$\|(T-\lambda I)v^{(k)}\| \geq \|Tv^{(k)}\| - \|\lambda v^{(k)}\| = (1-|\lambda|)\|v^{(k)}\| = (1-|\lambda|).$$

Thus, $\|(T-\lambda I)v^{(k)}\| \not\to 0$ as $1-|\lambda| > 0$.

b) The adjoint operator is uniquely determined by the property that $\langle T^*v, w\rangle = \langle v, Tw\rangle$ for all $v, w \in \ell^2(\mathbb{N})$. One quickly checks that this property is satisfied by the operator

$$T^* : x \in \ell^2(\mathbb{N}) \mapsto (x_2, x_3, \ldots).$$

The discrete spectrum of this operator has already been computed in FA I, Sheet 3 and is equal to the open unit ball in $\mathbb{C}$.

c) We claim first that $\operatorname{im}(T-\lambda I)^\perp = \ker(T^*-\overline{\lambda}I)$ for any $\lambda \in \mathbb{C}$. Indeed, $T^*-\overline{\lambda}I = (T-\lambda I)^*$ and so this follows from FA I, Sheet 13, 10b).

Note also that $\sigma_{\text{disc}}(T) = \emptyset$. The claim implies that

$$\overline{\operatorname{im}(T-\lambda I)} = \ker(T^*-\overline{\lambda}I)^\perp$$

and so for any $\lambda \notin \sigma_{\text{disc}}(T)$ we have $\lambda \in \sigma_{\text{resid}}(T)$ if and only if $\ker(T^*-\lambda I) \neq \{0\}$ if and only if $\overline{\lambda} \in \sigma_{\text{disc}}(T^*)$. We thus obtain the claim of part c).

5. **a)** If G is compact, we know from Theorem 3.47 that there are at most countably characters on G. Thus, $\widehat{G}$ is countable. We claim this implies that $\widehat{G}$ is discrete. Indeed, suppose that $\widehat{G}$ is not discrete and let m be the Haar measure on $\widehat{G}$. Then any point in $\widehat{G}$ has m-measure zero (see Sheet 8). Since $\widehat{G}$ is countable, this also yields that $\widehat{G}$ has measure zero which is impossible as $\widehat{G}$ is open and non-empty (there is the trivial character).

b) Assume that G is discrete and let m_G be the Haar measure on G. Then $L^1_{m_G}(G) = L^1(G)$ is unital (Sheet 8) and so $\widehat{G}$ is compact as $\sigma(L^1(G))$ is compact by Theorem 11.23.

c) Suppose that G is not discrete. Then any point set $\{g\}$ for $g \in G$ is not open as otherwise any other point set would also be open (using translations). Let $U \subset G$ be any open set containing the identity $e \in G$. Then there is $g \in U \setminus \{e\}$. Since characters separate points, there must be some $\chi' \in \widehat{G}$ such that $\chi'(g) \neq \chi'(e) = 1$. By the geometry of the circle, we can take a power n of $\lambda = \chi'(g)$ such that λ^n is in the closed left-half of the circle $\mathbb{S}^1$. So setting $\chi = (\chi')^n$ we get that $|\chi(g) - 1| \geq 1$.

To conclude that $\widehat{G}$ is non-compact, we consider the open sets $U_n = B_{\frac{1}{n}}(G)$. By the above we find a sequence $g_n \in G$ and $\chi_n \in \widehat{G}$ such that $g_n \to e$ and $|\chi_n(g_n) - 1| \geq 1$. If $\chi_{n_k} \to \chi$ along some subsequence $(n_k)_k$, then using the compact-open topology on $\widehat{G}$ we get that $\chi_{n_k}(g_{n_k}) \to \chi(e) = 1$ as $g_{n_k} \to e$. On the other hand, taking the limit in $|\chi_{n_k}(g_{n_k}) - 1| \geq 1$ that $|\chi(e) - 1| \geq 1$ which is impossible of course. Thus, the sequence χ_n has no convergent subsequence and the space $\widehat{G}$ cannot be compact (note that $\widehat{G}$ is metrizable).

6. **a)** We consider the Hilbert space $\ell^2(\mathbb{N})$ and the operator

$$\begin{aligned} T : v \in \ell^2(\mathbb{N}) \mapsto & T(v) \\ & (T(v))_n = \tfrac{1}{n} v_n. \end{aligned}$$

One checks that the discrete spectrum in this case is given by

$$\sigma_{\text{disc}}(T) = \{\tfrac{1}{n} : n \in \mathbb{N}\}$$

which is in particular not closed.

b) Let $(\lambda_n)_n$ be a sequence in $\sigma_{\text{appt}}(T)$ and suppose that $\lambda_n \to \lambda \in \mathbb{C}$. We need to show that $\lambda \in \sigma_{\text{appt}}(T)$. Let $\varepsilon > 0$. Then there is some λ_n with $|\lambda_n - \lambda| < \frac{\varepsilon}{2}$. Since $\lambda_n \in \sigma_{\text{appt}}(T)$ we may choose a vector $v \in \mathcal{H}$ of norm one such that $\|(T - \lambda_n)v\| < \frac{\varepsilon}{2}$. We then obtain

$$\begin{aligned} \|(T - \lambda)v\| &= \|(T - \lambda_n)v + (\lambda_n - \lambda)v\| \leq \|(T - \lambda_n)v\| + \|(\lambda_n - \lambda)v\| \\ &< \tfrac{\varepsilon}{2} + |\lambda_n - \lambda| < \varepsilon. \end{aligned}$$

This proves the claim.

c) As in the hint, we consider the Hilbert space $\mathcal{H} = L^2([0,1])^2$ (where we use the Lebesgue measure) and the operator $T : (f,g) \in \mathcal{H} \mapsto (M_I f, f) \in \mathcal{H}$. Here, the multiplication operator is as usual given by $M_I f(x) = x f(x)$. Let us begin by computing the discrete spectrum and the eigenspaces attached to any eigenvalue.

If $\lambda \in \sigma_{\text{disc}}(T)$ then there is a non-zero vector $(f,g) \in \mathcal{H}$ with

$$(T - \lambda)(f,g) = (M_I f - \lambda f, f - \lambda g) = 0.$$

Then $M_I f(x) = x f(x) = \lambda f(x)$ almost everywhere and so $f = 0$. On the other hand, this shows that $\lambda g = 0$. So $\lambda = 0$ as $(f,g) \neq 0$. Since $T(0,g) = 0$ for any g we have just proven that $\sigma_{\text{disc}}(T) = \{0\}$ and $\ker(T) = \{0\} \times L^2([0,1])$.

We claim that $\sigma_{\text{appt}}(T) = [0,1]$ and that $0 \notin \sigma_{\text{approx}}(T)$. Together with the equality $\sigma_{\text{appt}}(T) = \sigma_{\text{approx}}(T) \cup \sigma_{\text{disc}}(T)$ these two claims show that $\sigma_{\text{approx}}(T) = (0,1]$, which is in particular not closed.

For the first claim, notice that $\ker(\mathrm{T})^{\perp} = L^2([0,1]) \times \{0\}$. Assume that $\varepsilon > 0$ is small and that $(f,0) \in \mathcal{H}$ is a vector of norm one with the property that $\|T(f,0)\| < \varepsilon$. Then

$$\|T(f,0)\|^2 = \|M_I f\|_{L^2}^2 + \|f\|_{L^2}^2 < \varepsilon^2$$

and so in particular $\|(f,0)\| = \|f\|_{L^2} < \varepsilon$ which is a contradiction.

It remains to prove that $\sigma_{\mathrm{appt}}(T) = [0,1]$. Let $\lambda \in (0,1]$. We consider the interval $U = [\lambda - \varepsilon, \lambda + \varepsilon] \cap [0,1]$ and the functions $f = \mathbb{1}_U$, $g = \frac{1}{\lambda} f$. Then

$$\|(f,g)\|^2 = \|f\|_{L^2}^2 + \|g\|_{L^2}^2 = \big(1 + \tfrac{1}{\lambda^2}\big)|U|$$

where $|U|$ denotes the length of the interval. Note that $|U| \in [\varepsilon, 2\varepsilon]$ so that the norm of (f,g) is comparable to $\sqrt{\varepsilon}$. By definition, we have $T(f,g) = (M_I f - \lambda f, 0)$. Note that as f is supported on U

$$|(M_I f - \lambda f)(x)| = |x - \lambda||f(x)| \leq \varepsilon |f(x)|$$

and so

$$\|(T - \lambda)(f,g)\|^2 = \|M_I f - f\|_{L^2}^2 \leq \varepsilon^2 \|f\|_{L^2}^2.$$

Taking the square-root and dividing by the norm of (f,g) we get that $v = \frac{1}{\|(f,g)\|}(f,g)$ satisfies

$$\|(T - \lambda)v\| \leq \varepsilon \frac{\|f\|_{L^2}}{\|(f,g)\|} = \varepsilon\big(1 + \tfrac{1}{\lambda^2}\big).$$

Since λ is fixed, the right hand side can be made arbitrarily small and so $\lambda \in \sigma_{\mathrm{appt}}(T)$.

For the converse we assume that $\lambda \notin [0,1]$ and show that $\lambda \notin \sigma_{\mathrm{appt}}(T)$. We let $\delta > 0$ such that $|\lambda - x| > \delta$ for all $x \in [0,1]$. For $(f,g) \in \mathcal{H}$ a unit vector this implies

$$\|M_I f - \lambda f\| \geq \delta \|f\|_{L^2}.$$

If we now assume that there is a sequence $(f_n, g_n) \in \mathcal{H}$ of unit vectors with $\|(T - \lambda)(f_n, g_n)\| \to 0$ then the above lower bound proves that $f_n \to 0$. On the other hand, we know that the second component of $(T - \lambda)(f_n, g_n)$ is equal to $f_n - \lambda g_n$ and since λ is non-zero, we must also have $g_n \to 0$. This however is impossible.