Prof. Dr. Manfred Einsiedler
Andreas Wieser

Solutions for exercise sheet 11

1. By construction of the spectral measures we have

$$\langle av, w\rangle = \int_{\sigma(\mathcal{A})} a^\circ \,\mathrm{d}\mu_{v,w} \tag{1}$$

for all $a \in \mathcal{A}$ and that this property determines the spectral measure uniquely.

Now if $\langle v\rangle_{\mathcal{A}} \perp \langle w\rangle_{\mathcal{A}}$ then in particular $\langle av, w\rangle = 0$ for all $a \in \mathcal{A}$ and so the measure $\mu = 0$ satisfies (1). By uniqueness, $\mu_{v,w} = \mu = 0$.

Conversely, if $\mu_{v,w} = 0$ then by (1) we have $\langle av, w\rangle = 0$ for all $a \in \mathcal{A}$. Applying this to $a = a_1^* a_2$ for $a_1, a_2 \in \mathcal{A}$ (recall that $\mathcal{A}$ is a C^*-algebra) we get

$$\langle a_2 v, a_1 w\rangle = \langle a_1^* a_2 v, w\rangle = 0$$

and thus $\langle w\rangle_{\mathcal{A}} \perp \langle v\rangle_{\mathcal{A}}$.

2. **a)** Let μ be the measure with $\mathrm{d}\mu = |f|^2 \,\mathrm{d}\mu_v$. By uniqueness we may show that μ satisfies the defining property of the spectral measure (see also (1)). So let $a \in \mathcal{A}$ and let $\psi : \mathcal{H} \to L^2_{\mu_v}(\sigma(\mathcal{A}))$ be a unitary isomorphism as in the spectral theorem. Then

$$\langle aw, w\rangle = \langle \psi(aw), \psi(w)\rangle_{L^2} = \langle a^\circ f, f\rangle_{L^2} = \int_{\sigma(\mathcal{A}} a^\circ f \overline{f} \,\mathrm{d}\mu_v = \int_{\sigma(\mathcal{A})} f \,\mathrm{d}\mu$$

and so the claim follows.

b) Suppose that ϕ as in the exercise exists. Let $u = \phi(w)$. Then

$$\langle au, u\rangle = \left\langle \phi^{-1}(au), u\right\rangle = \langle aw, w\rangle .$$

This shows that the spectral measure μ_u, μ_w are equal again by uniqueness of the measure that satisfies the characterising property. But $u \in \langle v\rangle_{\mathcal{A}}$ and so the statement follows from a).

Conversely, suppose that $\mu_w \ll \mu_v$ and let g be the Radon-Nikodym derivative. Since $g \in L^1_{\mu_v}(\sigma(\mathcal{A}))$ is positive as μ_w is a positive measure, we may set $f =$

$\sqrt{g} \in L^2_{\mu_v}(\sigma(\mathcal{A}))$. Let $u \in \langle v \rangle_{\mathcal{A}}$ be the vector corresponding to f. Then by part a) $\mathrm{d}\mu_u = |f|^2 \,\mathrm{d}\mu_v = \mathrm{d}\mu_w$. Letting ψ_u resp. ψ_w be the unitary isomorphisms from the spectral theorem we consider the unitary isomorphism

$$\phi = \psi_u^{-1} \circ \psi_w : \langle w \rangle_{\mathcal{A}} \to \langle u \rangle_{\mathcal{A}}.$$

It satisfies

$$\phi(aw') = \psi_u^{-1}(a^\circ \psi_w(w')) = a\psi_u^{-1} \circ \psi_w(w') = a\phi(w')$$

for any $w' \in \langle w \rangle_{\mathcal{A}}$ and $a \in \mathcal{A}$. Thus, we are done.

3. a) Notice that the map

$$\iota : \chi \in \sigma(\mathcal{A}_T) \mapsto \chi(T) \in \sigma_{\mathcal{A}_T}(T)$$

is continuous as the topology on $\sigma(\mathcal{A}_T)$ is the weak*-topology. We need to prove that ι is bijective as continuity of the inverse than follows from the fact that $\sigma(\mathcal{A}_T)$ is compact.

Assume that $\chi_1(T) = \chi_2(T)$ for some χ_1, χ_2. Then

$$\chi_1(T^*) = \overline{\chi_1(T)} = \overline{\chi_2(T)} = \chi_2(T^*)$$

and so by linearity and multiplicativity χ_1, χ_2 coincide on the dense subspace $\langle T^m (T^*)^n : m, n \in \mathbb{N}_0 \}$. By continuity, $\chi_1 = \chi_2$.

Surjectivity follows from Theorem 11.23.

b) Notice that for $\lambda \in \mathbb{C}$ the assumption that $T - \lambda I$ is invertible in $\mathcal{A}_T$ implies that it is invertible in $B(\mathcal{H})$. Thus, $\sigma_{B(\mathcal{H})}(T) \subset \sigma_{\mathcal{A}_T}(T)$. We now prove the converse inclusion.

We begin directly without the assumption that $\mathcal{H}$ is cyclic. We invoke Theorem 12.60 to find a finite measure μ on $X = \sigma(\mathcal{A}) \times \mathbb{N}$ so that there is a unitary isomorphism $\mathcal{H} \cong L^2_\mu(X)$ under which applying $a \in \mathcal{A}_T$ corresponds to multiplying with a° in each copy of $\sigma(\mathcal{A}_T)$ in X separately. By part a),

$$\iota : X \ni (\chi, n) \mapsto (\chi(T), n) \in \sigma_{\mathcal{A}_T}(T) \times \mathbb{N} =: X'$$

is a homeomorphism. The continuous function $T^\circ : (\chi, n) \mapsto \chi(T)$ on the left-hand side corresponds (tautologically) to the continuous function $g : (z, n) \mapsto z$ on the right-hand side. Thus,

$$\mathcal{H} \cong L^2_{\mu'}(X')$$

where μ' is the push-forward under ι and where multiplication by T on $\mathcal{H}$ corresponds to multiplication by g on X'. As usual, we denote by M_g the latter map. Let $\mathcal{H}' = L^2_{\mu'}(X')$. In particular, $\sigma_{B(\mathcal{H})}(T) = \sigma_{B(\mathcal{H}')}(M_g)$. By Exercise 6 in Sheet 8, the latter is equal to the essential range of g i.e. the set of $\lambda \in g(X') = \sigma_{\mathcal{A}_T}(T)$ so that for any open neighborhood U of λ we have that

$$g^{-1}(U) = U \times \mathbb{N}$$

has positive μ'-measure.

So suppose by contradiction that there is a non-empty open set $U \subset \sigma_{\mathcal{A}_T}(T)$ with $\mu'(U \times \mathbb{N}) = 0$. Then $\mu(\iota^{-1}(U \times \mathbb{N})) = \mu'(\iota^{-1}(U) \times \mathbb{N}) = 0$. Then there is a non-trivial continuous function $f \in C(\sigma(\mathcal{A}_T))$ with $f \geq 0$ and $f(x) = 0$ for all $x \notin \iota^{-1}(U)$. Since $C(\sigma(\mathcal{A}_T)) \cong \mathcal{A}_T$ we have $f = a^\circ$ for some $a \in \mathcal{A}_T$. Since a° is supported on U, we have that $M_{a^\circ} : L^2_\mu(X) \to L^2_\mu(X)$ is trivial. Thus, a is trivial as $\mathcal{H} \cong L^2_\mu(X)$, which is a contradiction.

c) As in the proof of Theorem 12.60 it is sufficient to consider the cyclic case. So let $v \in \mathcal{H}$ be such that $\mathcal{H} = \langle v \rangle_{\mathcal{A}}$. By the spectral theorem we know that $\mathcal{H}$ is unitarily isomorphic to $L^2_{\mu_v}(\sigma(\mathcal{A}_T))$ where applying a° corresponds to multiplying with a°. Since $\sigma(\mathcal{A}_T) \cong \sigma(T)$, we may consider the push-forward μ'_v of μ_v to $\sigma(T)$ under ι. This yields a unitary isomorphism $L^2_{\mu_v}(\sigma(\mathcal{A}_T)) \cong L^2_{\mu'_v}(\sigma(T))$.

We now determine what multiplication by a° corresponds to $L^2_{\mu'_v}(\sigma(T))$ for $a = T^m(T^*)^n$. For this, it suffices to compute the continuous function f on $\sigma(T)$ corresponding to a°. Notice that for $\chi(T) \in \sigma(T)$

$$f(\chi(T)) = a^\circ(\chi) = \chi(a) = \chi(T^m(T^*)^n) = \chi(T)^m \chi(T^*)^m = \chi(T)^m \overline{\chi(T)}^n.$$

Thus, f is the restriction of the function $z^m \overline{z}^n$ to the spectrum $\sigma(T)$. In particular,

$$\langle T^m(T^*)^n v, v \rangle = \int_{\sigma(\mathcal{A}_T)} a^\circ \, \mathrm{d}\mu_v = \int_{\sigma(T)} z^m \overline{z}^n \, \mathrm{d}\mu'_v.$$

Also, $\mathcal{H} \cong L^2_{\mu'_v}(\sigma(T))$ where applying $T^m(T^*)^n$ corresponds to multiplying by $z^m \overline{z}^n$. This proves Theorem 12.33.

d) One can find a proof of (FC1)–(FC6) as stated in Section 12.3 in the book in Propositions 12.67 and 12.68. Here we will just find the proper setup and define the functional calculus

$$\mathscr{L}^\infty(\sigma(T)) \to B(\mathcal{H}).$$

For this, we first need to transfer the non-diagonal spectral measures. Indeed, by part a) and the same calculation as in c) there is for any $v, w \in \mathcal{H}$ a uniquely determined finite signed measure $\mu'_{v,w}$ on $\sigma(T)$ with the property that

$$\langle T^m(T^*)^n v, w \rangle = \int_{\sigma(T)} z^m \overline{z}^n \, \mathrm{d}\mu'_{v,w}. \tag{2}$$

The measure $\mu'_{v,w}$ corresponds to the measure $\mu_{v,w}$ under ι. Also, $\mathscr{L}^\infty(\sigma(\mathcal{A}_T)) \cong \mathscr{L}^\infty(\sigma(T))$ under ι which gives the measurable functional calculus

$$\mathrm{FC} : \mathscr{L}^\infty(\sigma(T)) \to B(\mathcal{H}).$$

by composition. By construction, it is uniquely determined by the analogous requirement as in Proposition 12.66. In particular, $\mathrm{FC}(z^m \bar{z}^n) = T^m (T^*)^n$ which proves (FC1) by linearity of FC.

The continuity bound in (FC2) is direct follows from the fact that $\mathscr{L}^\infty(\sigma(\mathcal{A}_T)) \cong \mathscr{L}^\infty(\sigma(T))$ is an isometry. Since this isometry maps continuous functions to continuous functions (ι is a homeomorphism), the other property in (FC2) also follows. (FC3) is clear since $\mathscr{L}^\infty(\sigma(\mathcal{A}_T)) \cong \mathscr{L}^\infty(\sigma(T))$ is an isomorphism of algebras. For the remaining properties we refer to Proposition 12.68.

4. Write $\mathcal{H} = \ell^2(\mathbb{Z})$ for simplicity.

a) One checks that T is self-adjoint by plugging in the definition. Let us compute the spectrum of S_r. For this, notice that S_r is unitary so that

$$\|(S - \lambda I)v\| \geq \|Sv\| - |\lambda|\|v\| = (1 - |\lambda|)\|v\|$$

proves that there is $|\lambda| < 1$ in the approximate point spectrum of S_r (see also Exercise 4, Sheet 10). Furthermore, since S_r is unitary, it is normal and hence the residual spectrum is empty. As the norm of S_r is one, $\sigma(S_r) \subset \overline{B_1(0)}$. Overall, we obtain $\sigma(S_r) \subset \mathbb{S}^1$. Let us check that any $\lambda \in \mathbb{S}^1$ belongs to the spectrum. For this, consider for $N \in \mathbb{N}$ the unit vector

$$v = \tfrac{1}{2N+1}(\ldots, 0, \lambda^{-N}, \lambda^{-N+1}, \ldots, \lambda^{N-1}, \lambda^N, 0, \ldots)$$

Then

$$(S_r - \lambda I)v = \tfrac{1}{2N+1}(-\lambda^{-N+1} e_{-N} + \lambda^N e_{N+1})$$

and thus $\|(S_r - \lambda I)v\| \leq \frac{2}{2N+1}$ as $|\lambda| = 1$. As $N \in \mathbb{N}$ was arbitrary, this proves that $\lambda \in \sigma(S_r)$. In summary, we showed that

$$\sigma(S_r) = \mathbb{S}^1.$$

Let us now prove that

$$\sigma(T) = \left\{ \tfrac{\lambda + \bar{\lambda}}{2} : \lambda \in \sigma(S_r) \right\}$$

Notice that $\frac{1}{2}(e^{i\vartheta} + e^{-i\vartheta}) = \cos(\vartheta)$ for any $\vartheta \in \mathbb{R}$ and so since the cosine is onto $[-1, 1]$ the claim would yield that $\sigma(T) = [-1, 1]$. To prove the claim, notice that the spectrum of T can be computed in either of the algebras

$$\mathcal{A}_T \subset \mathcal{A}_{S_r} \subset B(\mathcal{H}).$$

Since $\sigma_{\mathcal{A}_T}(T) = \sigma_{B(\mathcal{H})}(T)$ and inclusion of algebras induces reverse inclusion of the spectra by Exercise 3b), we may compute $\sigma(T) = \sigma_{\mathcal{A}_{S_r}}(T)$. Indeed,

$$\begin{aligned}\sigma(T) &= \{\chi(T) : \chi \in \sigma(\mathcal{A}_{S_r})\} = \{\tfrac{1}{2}(\chi(S_r) + \chi(S_r^*)) : \chi \in \sigma(\mathcal{A}_{S_r})\} \\ &= \{\tfrac{1}{2}(\chi(S_r) + \overline{\chi(S_r)}) : \chi \in \sigma(\mathcal{A}_{S_r})\}.\end{aligned}$$

Since $\sigma(S_r) = \{\chi(S_r) : \chi \in \sigma(\mathcal{A}_{S_r})\}$ this proves the claim. Notice that we used Theorem 11.23 twice.

b) Let us write $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ where

$$\begin{aligned}\mathcal{H}_+ &= \{x \in \mathcal{H} : x_{-n} = x_n \text{ for all } n \in \mathbb{N}\}, \\ \mathcal{H}_- &= \{x \in \mathcal{H} : x_{-n} = -x_n \text{ for all } n \in \mathbb{N}\}.\end{aligned}$$

We claim that $\mathcal{H}_+$ and $\mathcal{H}_-$ are cyclic and generated by $v_+ = e_0 \in \mathcal{H}_+$ and $v_- = e_1 - e_{-1} \in \mathcal{H}_-$ respectively. One first checks by direct calculation that $\mathcal{H}_+$ and $\mathcal{H}_-$ are invariant under T.

For v_+ notice that $Te_0 = \frac{1}{2}(e_1 + e_{-1})$,

$$\begin{aligned}T^2 e_0 &= \tfrac{1}{2}(Te_1 + Te_{-1}) = \tfrac{1}{2}\big(\tfrac{1}{2}(e_2 + e_0) + \tfrac{1}{2}(e_{-2} + e_0)\big) \\ &= \tfrac{1}{4}e_2 + \tfrac{1}{4}e_{-2} + \tfrac{1}{2}e_0.\end{aligned}$$

This shows that

$$\langle e_0, Te_0, T^2(e_0)\rangle = \langle e_0, \tfrac{1}{2}(e_1 + e_{-1}), \tfrac{1}{2}(e_2 + e_{-2})\rangle.$$

To iterate this, notice that for any $k \in \mathbb{N}$

$$T\big(\tfrac{1}{2}(e_k + e_{-k})\big) = \tfrac{1}{2}\big(\tfrac{1}{2}(e_{k+1} + e_{-(k+1)}) + \tfrac{1}{2}(e_{k-1} + e_{-(k-1)})\big).$$

Thus, it follows from induction that

$$\langle e_0, Te_0, T^2(e_0), \dots, T^k e_0\rangle = \langle e_0, \tfrac{1}{2}(e_1 + e_{-1}), \dots, \tfrac{1}{2}(e_k + e_{-k})\rangle.$$

Now notice that the subspace spanned by the vectors $e_k + e_{-k}$ for $k \in \mathbb{N}$ is dense in $\mathcal{H}_+$. Thus, the above claim obtained by induction yields that $v_+ = e_0$ is a generator of $\mathcal{H}_+$ and in particular, $\mathcal{H}_+$ is cyclic.

For v_- one computes similarly for any $k \in \mathbb{N}$

$$T\big(\tfrac{1}{2}(e_k - e_{-k})\big) = \tfrac{1}{2}\big(\tfrac{1}{2}(e_{k+1} - e_{-(k+1)}) + \tfrac{1}{2}(e_{k-1} - e_{-(k-1)})\big).$$

As again the subspace spanned by the vectors $e_k - e_{-k}$ for $k \in \mathbb{N}$ is dense in $\mathcal{H}_-$, v_- is a generator of $\mathcal{H}_-$.

It remains to compute the spectral measures for v_+ and v_-. As T is self-adjoint, we need to compute the uniques measures $\mu_\pm$ that satisfy for all $n \in \mathbb{N}_0$

$$\langle T^n v_\pm, v_\pm\rangle = \int_{[-1,1]} z^n \, \mathrm{d}\mu_\pm(z).$$

So let us compute the inner products as on the left hand side.

- For v_+ we have by an earlier calculation $\langle Tv_+, v_+\rangle = 0$ and $\langle T^2 v_+, v_+\rangle = \frac{1}{2}$. Also, notice that the coordinates appearing in $T^n v_+$ for odd n are all odd and thus $\langle T^n v_+, v_+\rangle = 0$ for all odd n. To compute the coefficients of T^n for $n = 2m$ even notice that $T = \frac{1}{2}(S_r + S_l)$ and since S_r and $S_l = S_r^{-1}$ commute

$$T^n = 2^{-n}(S_r + S_l)^n = 2^{-n}\sum_{k=0}^{n}\binom{n}{k} S_r^{k-(n-k)} = 2^{-n}\sum_{k=0}^{n}\binom{n}{k} S_r^{2k-n}.$$

In particular,

$$\begin{aligned}\langle T^n e_0, e_0\rangle &= 2^{-n}\sum_{k=0}^{n}\binom{n}{k}\left\langle S_r^{2k-n} e_0, e_0\right\rangle = 2^{-n}\sum_{k=0}^{n}\binom{n}{k}\langle e_{2k-n}, e_0\rangle \\ &= 2^{-n}\binom{n}{m}.\end{aligned}$$

Overall, we have shown that for every $n \in \mathbb{N}_0$

$$\langle T^n e_0, e_0\rangle = \tfrac{1}{2}\int_{-1}^{1} \sin(\pi x)^n \,\mathrm{d}x.$$

Thus, the spectral measure μ_+ is the pushforward of the normalized Lebesgue measure under the homeomorphism $x \mapsto \sin(\pi x)$. In explicit terms, we may substitute $u = \sin(\pi x)$ where $\mathrm{d}u = \pi\cos(\pi x)\,\mathrm{d}x = \pi\sqrt{1-u^2}\,\mathrm{d}x$ and obtain

$$\langle T^n e_0, e_0\rangle = \int_{-1}^{1} u^n \tfrac{1}{2\pi\sqrt{1-u^2}}\,\mathrm{d}u$$

for all $n \in \mathbb{N}_0$. Thus, $\mathrm{d}\mu_+ = \frac{1}{2\pi\sqrt{1-u^2}}\,\mathrm{d}u$.
- As in the first bullet, we compute for v_- and $n \in \mathbb{N}_0$ the coefficients

$$\langle T^n v_-, v_-\rangle = \langle T^n e_1, e_1\rangle - 2\langle T^n e_1, e_{-1}\rangle + \langle T^n e_{-1}, e_{-1}\rangle.$$

Since S_r and S_ℓ are unitary and commute with T we have

$$\langle T^n v_-, v_-\rangle = 2\langle T^n e_0, e_0\rangle - 2\langle T^n e_0, e_2\rangle.$$

The first term was computed in the first bullet. For the second we proceed in the same fashion to obtain $\langle T^n e_0, e_0\rangle$ if n is odd and otherwise for $n = 2m$

$$\begin{aligned}\langle T^n e_0, e_2\rangle &= 2^{-n}\sum_{k=0}^{n}\binom{n}{k}\left\langle S_r^{2k-n} e_0, e_2\right\rangle = 2^{-n}\sum_{k=0}^{n}\binom{n}{k}\langle e_{2k-n}, e_2\rangle \\ &= 2^{-n}\binom{n}{m+1}.\end{aligned}$$

Together with the first term we get for $n = 2m$

$$\langle T^n v_-, v_- \rangle = 2^{-n+1}\binom{n}{m} - 2^{-n+1}\binom{n}{m+1}.$$

From here on, one can find an explicit expression in exactly the same way as in the first bullet by applying the formula in the hint again.

5. **a)** By uniqueness we may show that $\mu_v + \mu_w$ satisfies the defining property of the spectral measure for μ_{v+w}. Indeed, we compute using orthogonality of the cyclic spaces

$$\langle a(v+w), v+w \rangle = \langle av, v \rangle + \langle aw, w \rangle = \int_{\sigma(\mathcal{A})} a^\circ \,\mathrm{d}\mu_v + \int_{\sigma(\mathcal{A})} a^\circ \,\mathrm{d}\mu_w$$
$$= \int_{\sigma(\mathcal{A})} a^\circ \,\mathrm{d}(\mu_v + \mu_w)$$

for all $a \in \mathcal{A}$ which proves this part of the exercise.

b) Let us write the Hilbert space

$$\mathcal{H} = \bigoplus_{i \in \mathbb{N}} \langle v_i \rangle_{\mathcal{A}} \tag{3}$$

as an orthogonal sum of cyclic spaces where $\|v_i\| = 1$ for all i. We then set

$$v = \sum_{i=1}^{\infty} 2^{-i} v_i \in \mathcal{H}$$

and claim that this vector does the job.

We begin by showing that the spectral measure of v is good for basis vectors. If $i \in \mathbb{N}$ is fixed, then one can write $v = v' + 2^{-i} v_i$ where $\langle v' \rangle_{\mathcal{A}} \perp \langle v_i \rangle_{\mathcal{A}}$. Thus, by part a)

$$\mu_v = \mu_{v'} + \mu_{2^{-i} v_i} = \mu_{v'} + 2^{-2i} \mu_{v_i}.$$

This shows that $\mu_{v_i} \ll \mu_v$.

If $w \in \mathcal{H}$ is fixed and we write $w = \sum_{i=1}^{\infty} w_i$ according to (3) then

$$\mu_{w_i} \ll \mu_{v_i} \ll \mu_v$$

where we first applied Exercise 2 and then the above construction.

To deduce from this the analogous statement for μ_w we show as in part a) that

$$\mu_w = \sum_{i=1}^{\infty} \mu_{w_i}.$$

Note that by construction $\sum_{i=1}^{\infty} \|\mu_{w_i}\| = \sum_{i=1}^{\infty} \|w_i\|^2 = \|w\|^2$ and so the sum is convergent. Also,

$$\langle aw, w \rangle = \sum_{i=1}^{\infty} \langle aw_i, w_i \rangle = \sum_{i=1}^{\infty} \int_{\sigma(\mathcal{A})} a^\circ \, \mathrm{d}\mu_{w_i} = \int_{\sigma(\mathcal{A})} a^\circ \, \mathrm{d}\Big(\sum_{i=1}^{\infty} \mu_{w_i} \Big)$$

which implies the claim.

By monotone convergence, we deduce that

$$\mu_w = \sum_{i=1}^{\infty} \mu_{w_i} \ll \mu_v$$

as desired.

6. a) We claim that the operator M_I is compact if and only if μ consists of countably many atoms i.e. $\mu = \sum_{i=1}^{\infty} a_i \delta_{\lambda_i}$ where $\lambda_i \in \mathbb{C}$ converge to zero.

To prove the claim, assume that M_I is compact and let $\lambda \in \operatorname{supp}(\mu)$ be non-zero. We prove that λ is an atom. By contradiction let $r_j > 0$ be a sequence of radii with $r_j \to 0$ for which

$$\mu(B_{r_1}(\lambda)) > \mu(B_{r_3}(\lambda)) > \mu(B_{r_3}(\lambda)) > \dots$$

is a strictly decreasing sequence of numbers (which are positive as λ is in the support of μ). Set for $j \in \mathbb{N}$

$$f_j = \tfrac{1}{\mu\left(B_{r_j}(\lambda) \setminus B_{r_{j+1}}(\lambda)\right)} \mathbb{1}_{B_{r_j}(\lambda) \setminus B_{r_{j+1}}(\lambda)}.$$

Then $\|f_j\| = 1$ and for r_j such that $|\lambda - \lambda'| < |\lambda|/2$ for all $\lambda' \in B_{r_j}(\lambda)$

$$\|M_I f_j\| \geq \tfrac{|\lambda|}{2} \|f_j\| = \tfrac{|\lambda|}{2}.$$

Since the functions $M_I f_j$ are supported on disjoint sets, they are also orthogonal and hence

$$\|M_I f_j - M_I f_{j'}\|^2 = \|M_I f_j\|^2 + \|M_I f_{j'}\|^2 \geq 2\left(\tfrac{|\lambda|}{2}\right)^2.$$

This shows that the sequence $M_I f_j$ in the closure of the image of the closed unit ball under M_I does not have a convergent subsequence. This is a contradiction. By a similar argument one shows that the support of μ intersected with the complement of any ball around zero is finite. This yields this direction of the claim.

The converse will not be used in b), so we refrain from proving it here. We refer to FA I on how to characterize compact subsets of $\ell^2(\mathbb{N})$.

b) We prove the following theorem:

> **Theorem:** Let $\mathcal{H}$ be a separable Hilbert space and let $T : \mathcal{H} \to \mathcal{H}$ be a compact normal operator. Then there is an orthonormal basis v_n of $\mathcal{H}$ consisting of eigenvectors of T. Furthermore, the eigenvalues λ_n satisfy $\lambda_n \to 0$.

To prove the Theorem, we decompose $\mathcal{H}$ into cyclic subspaces $\mathcal{H}_n$ for T (or the algebra $\mathcal{A}_T$ from Exercise 3) and apply Theorem 12.33 to the restriction $T|_{\mathcal{H}_n}$. We deduce that for any $n \in \mathbb{N}$ the Hilbert space $\mathcal{H}_n$ is unitarily isomorphic to $L^2_{\mu_n}(\mathbb{C})$ for some finite measure μ_n where applying T corresponds to multiplying by z. Since T is compact, so is $M_I : L^2_{\mu_n}(\mathbb{C}) \to L^2_{\mu_n}(\mathbb{C})$ and thus μ_n is atomic as in a). In particular, considering the L^2-basis consisting of Dirac functions supported on the atoms of μ_n we see that there is an orthonormal basis of eigenfunctions of M_I. Thus, the same applies to $T|_{\mathcal{H}_n}$.

We collect all the so obtain eigenfunctions for the different cyclic spaces and obtain an orthonormal basis $v_1, v_2, \ldots$ of normalized eigenvectors of $\mathcal{H}$. Let us denote by $\lambda_1, \lambda_2, \ldots$ the eigenvalues. It remains to show that $\lambda_n \to 0$. For this, consider for $r > 0$ the subspace

$$\mathcal{H}_r = \overline{\langle v_j : |\lambda_j| > r \rangle}.$$

Notice that $\|Tv_j\| \geq r$ for any $v_j \in \mathcal{H}_r$. By orthogonality any proper sequence in the collection $\{Tv_j : v_j \in \mathcal{H}_r\}$ thus has no convergent subsequence if the collection is infinite. Hence, $\mathcal{H}_r$ is finite-dimensional and there are only finitely many eigenvectors v_j with eigenvalue at least r in absolute value. This concludes the proof of the theorem.