Prof. Dr. Manfred Einsiedler
Andreas Wieser

Solutions for exercise sheet 12

1. We need to check that π_* is linear and that

$$\pi_*(f_1 * f_2) = \pi_*(f_1)\pi_*(f_2) \tag{1}$$

for all $f_1, f_2 \in L^1(G)$.

Let us begin with linearity. So let $\alpha \in \mathbb{C}$ and $f_1, f_2 \in L^1(G)$. Also, we let $v, w \in \mathcal{H}$. Then

$$\begin{aligned}\langle(\alpha\pi_*(f_1) + \pi_*(f_2))v, w\rangle &= \alpha\,\langle\pi_*(f_1)v, w\rangle + \langle\pi_*(f_2)v, w\rangle \\ &= \int_G (\alpha f_1(g) + f_2(g))\,\langle\pi_g(v), w\rangle \\ &= \langle\pi_*(\alpha f_1 + f_2)v, w\rangle\end{aligned}$$

which proves linearity.

For Equation (1) we also let $f_1, f_2 \in L^1(G)$ and $v, w \in \mathcal{H}$. Then

$$\begin{aligned}\langle\pi_*(f_1)\pi_*(f_2)v, w\rangle &= \int_G f_1(g)\,\langle\pi_g\pi_*(f_2)v, w\rangle\,\mathrm{d}m(g) \\ &= \int_G f_1(g)\,\langle\pi_*(f_2)v, \pi_{g^{-1}}w\rangle\,\mathrm{d}m(g) \\ &= \int_G f_1(g)\int_G f_2(h)\,\langle\pi_h v, \pi_{g^{-1}}w\rangle\,\mathrm{d}m(h)\,\mathrm{d}m(g) \\ &= \int_G f_1(g)\int_G f_2(h)\,\langle\pi_{gh}v, w\rangle\,\mathrm{d}m(h)\,\mathrm{d}m(g) \\ &= \int_G f_1(g)\int_G f_2(g^{-1}h')\,\langle\pi_{h'}v, w\rangle\,\mathrm{d}m(h')\,\mathrm{d}m(g) \\ &= \int_G \langle\pi_{h'}v, w\rangle\int_G f_1(g)f_2(g^{-1}h')\,\mathrm{d}m(g)\,\mathrm{d}m(h') \\ &= \int_G \langle\pi_{h'}v, w\rangle\, f_1 * f_2(h')\,\mathrm{d}m(h') \\ &= \langle\pi_*(f_1 * f_2)v, w\rangle\end{aligned}$$

where we set $h' = gh$. This yields the claim in (1).

2. One checks directly that M_g as defined on the sheet is a unitary operator using that $|\chi(g)| = 1$ for any $g \in G$. Also, if $g_1, g_2 \in G$ then for any $f \in L^2_\mu(\hat{G})$ and for any $\chi \in \hat{G}$ we have

$$\begin{aligned}(M_{g_1} M_{g_2} f)(\chi) &= \chi(g_1)(M_{g_2} f)(\chi) = \chi(g_1)\chi(g_2) f(\chi) \\ &= \chi(g_1 g_2) f(\chi) = M_{g_1 g_2} f(\chi)\end{aligned}$$

and so $M_{g_1} M_{g_2} = M_{g_1 g_2}$. It remains to prove that for any $f \in L^2_\mu(\hat{G})$ then map $g \in G \mapsto M_g f \in L^2_\mu(\hat{G})$ is continuous. For this, we may suppose as in the proof of Lemma 3.74 that $f \in C_c(\hat{G})$. Let $K = \operatorname{supp}(f)$ and note that for $g, h \in G$

$$\begin{aligned}\|M_g f - M_h f\|^2_{L^2} &= \int_{\hat{G}} |\chi(g) - \chi(h)|^2 |f(\chi)|^2 \,\mathrm{d}\mu(\chi) \\ &= \int_K |\chi(h^{-1} g) - 1|^2 |f(\chi)|^2 \,\mathrm{d}\mu(\chi).\end{aligned}$$

Recall that the compact-open topology on the Pontryagin dual $\hat{G}$ is given by the neighborhood basis at any $\chi_0 \in \hat{G}$ of the form

$$U_{L,\varepsilon}(\chi_0) = \{\chi : \|\chi - \chi_0\|_{L,\infty} < \varepsilon\}$$

for $\varepsilon > 0$ and $L \subset G$ compact. Whenever $L_1 \supset L_2 \supset \ldots$ is a sequence of compact neighborhoods of the identity with $\cap_{n \in \mathbb{N}} L_n = \{e\}$ we have

$$\bigcup_{n \in \mathbb{N}} U_{L_n,\varepsilon}(1) = \hat{G}$$

Here, 1 denotes the trivial character. As the subsets are increasing, we may choose $n \in \mathbb{N}$ such that $K \subset U_{L_n,\varepsilon}(1)$. In other words, there is a compact neighborhood of the identity $L = L_n$ in G with the property that $|\chi(g') - 1| < \varepsilon$ for all $g' \in L$ and all $\chi \in K$. Letting $U = hL^\circ$ we obtain an open neighborhood of h so that

$$|\chi(h^{-1} g) - 1| < \varepsilon$$

for all $g \in U$. In particular,

$$\|M_g f - M_h f\|^2_{L^2} \leq \varepsilon^2 \|f\|^2_{L^2}.$$

This proves the claim.

3. **a)** By Lemma 12.13 the spectrum is contained in the numerical range of T which by assumption is contained in $[0, \infty)$.

b) Consider the (continuous) bounded function $f : x \mapsto x^{\frac{1}{n}}$ on the compact set $\sigma(T) \subset [0, \infty)$ and set $S = \mathrm{FC}(f)$. By (FC3) on p. 445 we have

$$S^n = \mathrm{FC}(f^n) = \mathrm{FC}((x \in \sigma(T) \mapsto x)) = T$$

where we could for instance use (FC1) to obtain the last equality.

To see that S is positive note that we can apply the same trick again find an operator $\sqrt{S} \in B(\mathcal{H})$ with $(\sqrt{S})^{2n} = T$. By (FC3) we again have $\sqrt{S}^2 = S$ and $\sqrt{S}^* = \sqrt{S}$. Then for any $v \in \mathcal{H}$

$$\langle Sv, v \rangle = \left\langle \sqrt{S}^2 v, v \right\rangle = \left\langle \sqrt{S}v, \sqrt{S}v \right\rangle \geq 0$$

as claimed.

c) If M_g (or M_h) is positive, then the spectrum is contained in $[0, \infty)$. On the other hand, the spectrum is equal to the essential range of g (see Sheet 8, Exercise 6) and thus g is real-valued and positive (when adapted on a null-set). The same applies to h. But then $g^n = h^n$ implies $g = h$.

d) Notice that S' commutes with T as $T = (S')^n$. By (FC5) the operator S' then also commutes with S. We let

$$\mathcal{A} = \overline{\langle S^{k_1}(S')^{k_2} : k_1, k_2 \in \mathbb{N}_0 \rangle}.$$

Applying Theorem 12.60 to $\mathcal{A}$ we find a finite measure space (X, μ) and a unitary isomorphism $\mathcal{H} \cong L^2_\mu(X)$ so that applying $a \in \mathcal{A}$ corresponds to multiplication by some function $g_a \in L^\infty_\mu(X)$. Letting $g = g_S$ and $h = g_{S'}$ we obtain multiplication operators as in c) and are thus done.

4. **a)** We consider the self-adjoint operator $T = B^*B : \mathcal{H}_1 \to \mathcal{H}_1$. Note that T is positive as

$$\langle Tv, v \rangle = \langle B^*Bv, v \rangle = \|Bv\|^2$$

so that $\sigma(T) \subset [0, \infty)$. We want to show that the spectrum of T consists of a point. Suppose that the contrary holds. Then we may find two continuous, non-negative and non-zero functions $f_1, f_2 \in C(\sigma(T))$ such that $f_1 f_2 = 0$. Using the functional calculus for T we may set $T_1 = \mathrm{FC}(f) = f_1(T)$ and $T_2 = \mathrm{FC}(f_2) = f_2(T)$.

Now note that for any $g \in G$ the operator T commutes with $\pi_1(g)$ as

$$\begin{aligned} T\pi_1(g) &= B^*\pi_2(g)B = B^*\pi_2(g^{-1})^*B = \big(\pi_2(g^{-1})B\big)^*B = \big(B\pi_1(g^{-1})^*B \\ &= \pi_1(g)B^*B = \pi_1(g)T. \end{aligned}$$

In particular, $\pi_1(g)$ commutes with the functional calculus of T and thus $\pi_1(g)T_1 = T_1\pi_1(g)$. Hence, the subspace $V = \ker(T_1)$ is invariant. As T_1 is non-zero by (12.6) in the book, the subspace V is strictly contained in $\mathcal{H}_1$. Also, T_2 is non-trivial and as $T_1T_2 = \mathrm{FC}(f_1f_2) = 0$ we get that $T_2v \in V$ for all $v \in \mathcal{H}_1$ and in particular $V \neq \{0\}$. We have thus found a closed invariant subspace of $\mathcal{H}_1$ which is not $\{0\}$ or $\mathcal{H}_1$. This contradicts irreducibility.

We have thus shown that $\sigma(T) = \{\lambda\}$ for some $\lambda \geq 0$. We now apply the spectral theorem for T to find a finite measure μ on $X = \sigma(T) \times \mathbb{N}$ such that there is a unitary isomorphism $\mathcal{H}_1 \cong L^2_\mu(X)$ under which applying T corresponds to multiplying with z in each component. Since $\sigma(T) = \{\lambda\}$ this shows that any vector $v \in \mathcal{H}_1$ is an eigenvector and thus $T = B^*B = \lambda 1_{\mathcal{H}_1}$ as claimed.

b) We set

$$S_1 = \tfrac{1}{2}(B + B^*), \quad S_2 = \tfrac{1}{2\mathrm{i}}(B - B^*)$$

to obtain two self-adjoint operators. The argument in a) then applies (ignoring the statements about positivity) to show that $S_1 = \lambda_1 1_{\mathcal{H}_1}$ and $S_2 = \lambda_2 1_{\mathcal{H}_1}$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. In particular,

$$B = S_1 + \mathrm{i}S_2 = (\lambda_1 + \mathrm{i}\lambda_2)1_{\mathcal{H}_1}$$

which proves the claim in this case as well.

5. As mentioned in the hint, we will apply the proof of Corollary 12.81. Let $v \in \mathcal{H}$ and $G = \mathbb{R}^d$. Let $\mathcal{A} \subset B(\mathcal{H})$ be the separable commutative unital C^*-algebra obtained from π_*. By (12.15) we find a finite measure $\mu_v^{(1)}$ on $\sigma(\mathcal{A})$ uniquely determined by the property

$$\langle av, v\rangle = \int_{\sigma(\mathcal{A})} a^\circ \,\mathrm{d}\mu_v^{(1)}.$$

Using the continuous map $\iota : L^1(G) \oplus \mathbb{C} \to \mathcal{A}$ with dense image we get a map as in the proof of Corollary 12.81 a continuous map

$$\iota^* : \sigma(\mathcal{A}) \to \sigma(L^1(G) \oplus \mathbb{C})$$

which is a homeomorphism onto its image. Let $\mu_v^{(2)}$ be the push-forward of $\mu_v^{(1)}$ under ι so that we obtain a unitary isomorphism $\langle v\rangle_{\mathcal{A}} \cong L^2_{\mu_v^{(1)}}(\sigma(\mathcal{A}))$ then gives a unitary isomorphism

$$\langle v\rangle_{\mathcal{A}} \cong L^2_{\mu_v^{(2)}}(\sigma(L^1(G) \oplus \mathbb{C}))$$

under which applying $\pi_*(f)$ for $f \in L^1(G)$ on the left-hand side corresponds to multiplying with f° on the right-hand side (see Corollary 12.81). In particular,

$$\langle \pi_*(f)v, v\rangle = \int_{\sigma(L^1(G)\oplus\mathbb{C})} f^\circ \,\mathrm{d}\mu_v^{(2)}.$$

Recall that we may identify (by homeomorphisms)

$$\sigma(L^1(G)\oplus\mathbb{C}) \simeq \sigma(L^1(G)) \cup \{0\}, \quad \sigma(L^1(G)) \simeq \hat{G}.$$

Letting $\mu_v^{(3)}$ be the pushforward of $\mu_v^{(2)}$ to $\sigma(L^1(G)) \cup \{0\}$.

As in the proof of Corollary 12.81 we show that $\mu_v^{(3)}(\{0\}) = 0$. Suppose otherwise and let $w \in \mathcal{H}$ be the non-zero vector corresponding to $\mathbb{1}_{\{0\}}$. By continuity of the unitary representation there is some open neighborhood of the identity $0 \in G = \mathbb{R}^d$ such that $\mathrm{Re}(\langle gw, w\rangle) > 0$ for all $g \in B$. In particular,

$$\mathrm{Re}(\langle \pi_*(\mathbb{1}_B)w, w\rangle) = \int_B \mathrm{Re}(\langle \pi_g w, w\rangle)\,\mathrm{d}m(g) > 0$$

and $\langle \pi_*(\mathbb{1}_B)w, w\rangle$ is non-zero. On the other hand, by definition of w we have

$$\langle \pi_*(\mathbb{1}_B)w, w\rangle = \int_{\sigma(L^1(G))\cup\{0\}} \mathbb{1}_B^\circ \mathbb{1}_{\{0\}}\,\mathrm{d}\mu_v^{(3)} = 0$$

which is a contradiction.

We may thus define μ_v as the pushforward of $\mu_v^{(3)}$ to $\hat{G}$. Tracing through the proof of Corollary 12.81 then shows the claim of the exercise.

6. **a)** Let $f = \mathbb{1}_I$ where $I = [0,1]$. We claim that any $g \in L^2(\mathbb{R})$ with $\langle \pi_x f, g\rangle = 0$ for all $x \in \mathbb{R}$ is zero. This shows the claim of a) assuming that the two notions of cyclicity agree. We show that $L^2(\mathbb{R})$ is cyclic for the algebra below. So let $g \in L^2(\mathbb{R})$ with this property. We claim that

$$\langle \mathbb{1}_J, g\rangle = 0$$

for any (closed) interval $J \subset \mathbb{R}$. Write $J = [a,b]$. First, we observe that

$$\left\langle \pi_x \mathbb{1}_{[0,N]}, g\right\rangle = \sum_{k=0}^{N-1} \langle \pi_x \pi_k f, g\rangle = 0.$$

for any $N \in \mathbb{N}$ and $x \in \mathbb{R}$. In particular, for any $N \in \mathbb{N}$

$$\begin{aligned} 0 &= \left\langle \pi_a \mathbb{1}_{[0,N]} - \pi_b \mathbb{1}_{[0,N]}, g\right\rangle = \left\langle \mathbb{1}_{[a,N+a]} - \mathbb{1}_{[b,N+b]}, g\right\rangle \\ &= \left\langle \mathbb{1}_{[a,b]} - \mathbb{1}_{[N+a,N+b]}, g\right\rangle = \left\langle \mathbb{1}_{[a,b]}, g\right\rangle - \left\langle \mathbb{1}_{[N+a,N+b]}, g\right\rangle. \end{aligned}$$

Note that

$$
\begin{aligned}
|\left\langle \mathbb{1}_{[a,b]}, g \right\rangle|^2 = |\left\langle \mathbb{1}_{[N+a,N+b]}, g \right\rangle|^2 &\le \|\mathbb{1}_{[N+a,N+b]}\|^2 \|\mathbb{1}_{[N+a,N+b]} g\|^2 \\
&= |b-a| \|\mathbb{1}_{[N+a,N+b]} g\|^2 \\
&\le |b-a| \int_{N+a}^{\infty} |g(x)|^2 \, \mathrm{d}x
\end{aligned}
$$

and since the right-hand side gets arbitrarily small as $N \to \infty$, we have that $\left\langle \mathbb{1}_{[a,b]}, g \right\rangle = 0$. This proves our first claim. It implies that $g = 0$ almost everywhere, since almost every point is a Lebesgue point.

CYCLIC FOR THE ALGEBRA: Let f be as above and let $g \in L^2(\mathbb{R})$ satisfy $\langle af, g \rangle = 0$ for all $a \in \mathcal{A}$. We claim that this implies that $\langle \pi_x f, g \rangle = 0$ for all $x \in \mathbb{R}$ and thus $g = 0$. For this, we use the proof of Corollary 12.81: whenever $\psi_k = \frac{1}{m(B_k)} \mathbb{1}_{B_k}$ for a decreasing sequence of neighborhoods B_k of the identity $0 \in \mathbb{R}$ and $\psi_k^x = \psi_k(\cdot - x)$ then $\pi_*(\psi_k^x)v \to \pi_x v$ as $k \to \infty$ for any $v \in L^2(\mathbb{R})$. Thus, taking the limit as $k \to \infty$ of

$$
0 = \langle \pi_*(\psi_k^x) f, g \rangle
$$

shows the claim.

b) We first discuss the "toy" problem described in the hint.

"TOY" PROBLEM: So let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a bounded operator such that $T \circ M_\phi = M_\phi \circ T$ for any $\phi \in L^2(\mathbb{R})$.

Consider $f_n = \mathbb{1}_{[-n,n]}$. Then $Tf_n \in L^2(\mathbb{R})$ is supported on $[-n, n]$. Indeed, for any interval I intersecting $[-n, n]$ we have

$$
\mathbb{1}_I T f_n = M_{\mathbb{1}_I} T f_n = T M_{\mathbb{1}_I} f_n = T(\mathbb{1}_I \mathbb{1}_{[-n,n]}) = 0.
$$

We claim that $Tf_n \in \mathscr{L}^\infty([-n, n])$ and seek an explicit bounded on the supremum norm. Let $s > 0$ and set

$$
F_s = \{x \in [-n, n] : |Tf_n(x)| > s\}.
$$

Then

$$
\|T(f_n \mathbb{1}_{F_s})\|^2 = \|\mathbb{1}_{F_s} T(f_n)\|^2 = \int_{F_s} |Tf_n(x)|^2 \, \mathrm{d}x \ge s^2 m(F_s)
$$

and thus $\|T(f_n \mathbb{1}_{F_s})\| \ge s\sqrt{m(F_s)}$. On the other hand,

$$
\|T(f_n \mathbb{1}_{F_s})\| = \|T(\mathbb{1}_{F_s})\| \le \|T\| \|\mathbb{1}_{F_s}\| = \|T\| \sqrt{m(F_s)}.
$$

Overall, we obtain that $s \le \|T\|$ whenever $m(F_s)$ is non-zero. In particular,

$$
\|Tf_n\|_\infty \le \|T\|.
$$

We define $g \in \mathscr{L}^\infty(\mathbb{R})$ by setting $g(x) = Tf_n(x)$ whenever $x \in [-n, n]$. We automatically have $\|g\|_\infty \leq \|T\|$ and only need to check that g is well-defined. So if $x \in [-n, n] \subset [-m, m]$ then we can write

$$f_m = \mathbb{1}_{[-m,-n]} + f_n + \mathbb{1}_{[n,m]}.$$

Then by the previous argument $T\mathbb{1}_{[-m,-n]}$ is supported on $[-m, -n]$ and similarly for $T\mathbb{1}_{[n,m]}$ which shows that (ignoring null-sets)

$$Tf_m(x) = T\mathbb{1}_{[-m,-n]}(x) + Tf_n(x) + T\mathbb{1}_{[n,m]}(x) = Tf_n(x)$$

and hence g is well-defined.

For our "toy" case it remains to check that $T = M_g$. For this, we can check that T and M_g agree for functions of the kind $\mathbb{1}_I$ where I is a bounded interval as these functions are dense in L^2. Let $n \in \mathbb{N}$ be large enough such that $I \subset [-n, n]$. Then

$$T\mathbb{1}_I = T(\mathbb{1}_I f_n) = \mathbb{1}_I T(f_n) = \mathbb{1}_I g f_n = \mathbb{1}_I g = M_g(\mathbb{1}_I)$$

as claimed.

FULL PROBLEM: Let $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be such that $\pi_x T = T\pi_x$ for all $x \in \mathbb{R}$. We show that $T = \mathrm{FC}(\psi)$ for some $\psi \in \mathscr{L}^\infty(\mathbb{R})$.

We need to make sense of what we mean by functional calculus in this case. By Exercise 5, part a) in the present exercise and Corollary 12.81 there is a finite measure μ' on $\hat{\mathbb{R}}$ and a unitary isomorphism $L^2(\mathbb{R}) \cong L^2_{\mu'}(\hat{\mathbb{R}})$ such that applying π_x on $L^2(\mathbb{R})$ corresponds to multiplying with the function $\chi \in \hat{\mathbb{R}} \mapsto \chi(x)$. Since for the measure μ' (bounded) functions on $\sigma(\mathcal{A})$ correspond to (bounded) functions on $\hat{\mathbb{R}}$ we obtain a functional calculus

$$\mathrm{FC} : L^\infty_{\mu'}(\hat{\mathbb{R}}) \to B(L^2(\mathbb{R}))$$

Identifying $\mathbb{R}$ with $\hat{\mathbb{R}}$ under $t \mapsto \chi_t$ (see the remark in 5) we obtain a finite measure μ on $\mathbb{R}$ and a unitary isomorphism $L^2(\mathbb{R}) \cong L^2_\mu(\mathbb{R})$ so that applying π_x corresponds to multiplying with the function $t \mapsto \chi_t(x) = \mathrm{e}^{2\pi \mathrm{i}\langle t,x\rangle}$. We also obtain the functional calculus

$$\mathrm{FC} : L^\infty_\mu(\mathbb{R}) \to B(L^2(\mathbb{R})).$$

COMMUTING WITH THE UNITARY REPRESENTATION: Since T commutes with all π_x for $x \in \mathbb{R}$ we would expect T to commute with the functional calculus. We verify this here. Note that by (FC5) it suffices to show that T commutes with any

$a \in \mathcal{A}$ or equivalently that T commutes with $\pi_*(f)$ for any $f \in L^1(G)$. To this end, we compute for $v, w \in \mathcal{H}$

$$\begin{aligned}\langle T\pi_*(f)v, w\rangle &= \langle \pi_*(f)v, T^*w\rangle = \int_{\mathbb{R}} f(x)\, \langle \pi_x v, T^* w\rangle \,\mathrm{d}x \\ &= \int_{\mathbb{R}} f(x)\, \langle T\pi_x v, w\rangle \,\mathrm{d}x = \int_{\mathbb{R}} f(x)\, \langle \pi_x T v, w\rangle \,\mathrm{d}x \\ &= \langle \pi_*(f)Tv, w\rangle\end{aligned}$$

as desired.

APPLYING THE UNITARY ISOMORPHISM: As above, we let μ be a finite measure on $\mathbb{R}$ so that there is a unitary isomorphism $\mathcal{H} \cong L^2_\mu(\mathbb{R})$ under which applying π_x corresponds to multiplying with $t \mapsto \mathrm{e}^{2\pi\mathrm{i}\langle x,t\rangle}$. Let T' be the operator on $L^2_\mu(\mathbb{R})$ corresponding to T. Then T' commutes with any M_f when $f \in L^\infty_\mu(\mathbb{R})$. We generalize the above argument to show that $T' = M_\psi$ for some $\psi \in L^\infty_\mu(\mathbb{R})$. This concludes that $T = \mathrm{FC}(\psi)$ and hence the exercise.

To do this, we take some care about null-sets. Note that the map $a > 0 \mapsto \mu([-a, a])$ is monotonely increasing and thus continuous at all but countably many points. Let $A \subset \mathbb{R}$ be this set of continuity points and note that $\mu(\{a\}) = \mu(\{-a\}) = 0$ for all $a \in A$. The argument in the toy problem then applies when one considers the functions $f_a = \mathbb{1}_{[-a,a]}$ for $a \in A$.