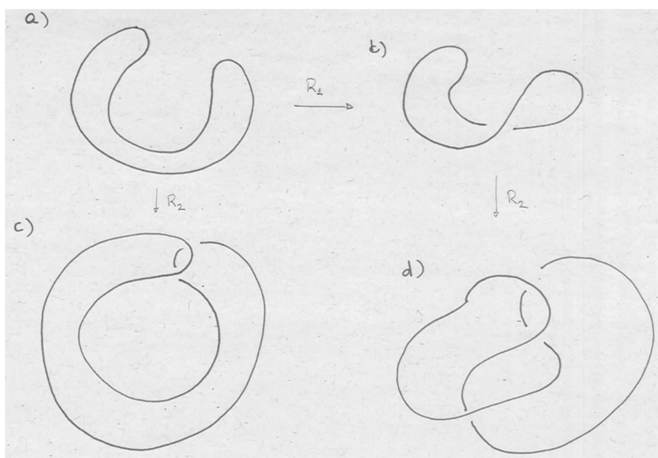
Exercise sheet 11: Seifert surfaces

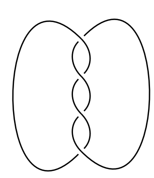
1. Show that the construction of the Seifert surface does not depend on the chosen orientation of the knot.

2. (a) Consider the standard projection of the trefoil knot and construct a surface bounding the knot using Seifert's algorithm. Classify the resulting surfaces and compare your result with the bounding surface we obtained in the lecture for the trefoil using a chessboard colouring.
   (b) Consider the standard projection of the figure-eight knot and construct a surface bounding the knot using a chessboard colouring. Classify the resulting surfaces and compare your result with the bounding surface we obtained in the lecture for the figure-eight knot using Seifert's algorithm.

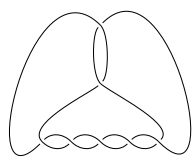
3. Construct the Seifert surface and compute the genus for the following four diagrams of the unknot.

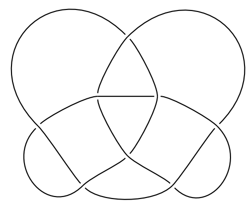


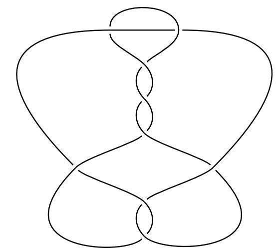
4. Consider the following 2-component link. Depending on your choice of orientations you can construct two different Seifert surfaces using Seifert's algorithm. Do this, compute their genera and decide what the genus of the link ought to be.



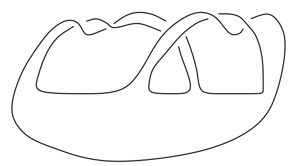
5. Use Seifert's algorithm to find surfaces bounding the following knots and use the Euler characteristic to identify these surfaces.







6. Consider the knot in question 7 on sheet 10 again and apply Seifert's algorithm to construct a surface bounded by this knot. What is the genus of this surface?



7. Express the genus of the Seifert surface in terms of the number of Seifert circles (or circuits) s and the number of crossings c in the knot diagram.

8. Viewing the Seifert surface constructed from Seifert's algorithm as a disc-and-band surface, prove the following upper bound for the genus of a knot:
$$g(K) \leq \frac{c(K)}{2}$$

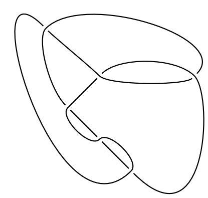
9. Prove that any genus-1 knot is prime.

10. The following statement has been proved by various people, the probably easiest proof is due to David Gabai: *Applying Seifert's algorithm to an alternating projection of an alternating knot or link does yield a Seifert surface of minimal genus.*

    (a) Use this statement to compute the genus of the knots 6_3 and 7_6.

    (b) What does this tell you about the reverse of question 9?

11. Consider the following knot diagram:



    (a) Use Seifert's algorithm to construct a Seifert surface for this knot diagram and identify it using the theorem on classification of surfaces.
    (b) Compute the genus of this knot and decide whether it is prime.

12. Use the genus to prove that there are infinitely many different knots.

13. Let K be a non-trivial knot. Prove that there does not exist a knot J such that the connected sum $K\#J$ is trivial i.e. prove that you cannot "unknot" a knot by adding another knot or in other words that knots do not have inverses.

Due Date: 20.05.2019