Exercise sheet 2: Formal introduction to knots

1. Show that a knot diagram with n crossings can be turned into a diagram of the unknot by replacing at most $\lfloor \frac{n-1}{2} \rfloor$ overcrossings by undercrossings.

2. Show that any two embeddings $f_1, f_2 : S^1 \hookrightarrow \mathbb{R}^3$ are homotopic.

3. Show that

   $$K \sim^{\Delta} K' \Leftrightarrow \{K' \text{ is obtained by } K \text{ using a finite sequence of } \Delta\text{- and } \Delta^{-1}\text{-moves}\}$$

   defines an equivalence relation.

4. (a) Check that you can move one of the vertices of a knot a very small distance, keeping the rest fixed, by using two successive Δ-moves.

   (b) Check that you can effectively add a new vertex in the middle of any edge by using three Δ-moves.

5. Show that for every polygonal knot $K \subset \mathbb{R}^3$ there exists $\varepsilon > 0$ such that every ε-perturbation K' is equivalent to K.

6. (a) Let $P \subset \mathbb{R}^2$ be a simple closed polygonal curve. Show that the complement $\mathbb{R}^2 \backslash P$ has two components, one bounded and one unbounded.

   (b) Show that the closure of the bounded component is homeomorphic to a closed disk.

7. (a) Show that it is possible to represent the trefoil by a polygon with 6 points $(p_1, \dots, p_6)$ but that it is not possible to construct a non-trivial knot with fewer than 6 points.

   (b) Show that you can represent the knot 5_1 as a polygon with 8 points.

8. Let $(p_1, \dots, p_n)$ be an ordered n-tuple of points in $\mathbb{R}^3$ defining a piecewise linear knot K.

   (a) Find an example where a permutation of the points $p_1, \dots p_n$ is still a closed polygon but no longer a simple polygon.

   (b) Show that a permutation of the points $\{p_i\}$ of the trefoil (from ex. 7a)) can define a knot which is equivalent to the unknot.

Due Date: 04.03.2019