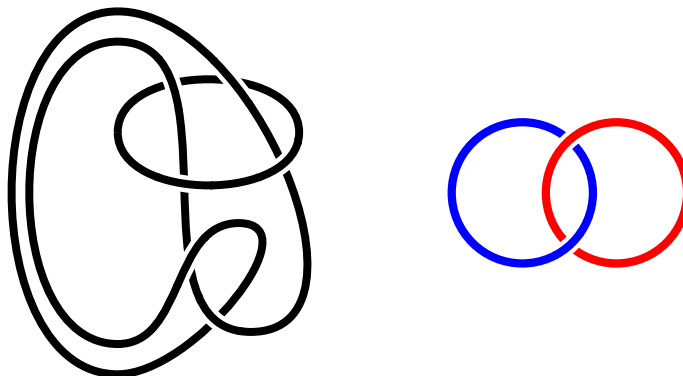


Exercise sheet 4: p -colourings

1. Prove that any knot diagram has at least three 3-colourings and hence $\tau_3(D) \geq 3$.
2. Show that $\tau_3(L)$ is invariant under Reidemeister move 3.
3. Compute τ_3 of the unknot and $\tau_3(5_1)$ – what do you observe? Can we use τ_3 to distinguish the two knots with crossing number 5?
4. Show that the linking number fails to distinguish the Whitehead link from the unlink, but that τ_3 succeeds.
5. (a) Show that $\tau_3(K)$ is always divisible by 3.
 (b) Show that $\tau_3(K)$ is always a power of 3.
 (c) Show that $\tau_3(K) - 3$ is always divisible by 3!
6. Show that τ_3 cannot distinguish a knot from its mirror image.
7. Use $\tau_3(K_1 \# K_2) = \frac{1}{3} \tau_3(K_1) \tau_3(K_2)$ to prove that there are infinitely many distinct knots.
8. Calculate $\tau_3(L)$ for the following two two-component links. Can you decide whether these links are equivalent or not?



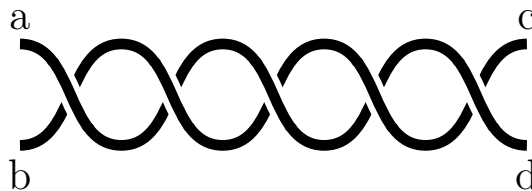
9. Determine for each of the following statements whether it is true or false.
 - (a) If K_1 and K_2 are knots with $\tau_3(K_1) = \tau_3(K_2)$ then K_1 is equivalent to K_2 .
 - (b) If K_1 and K_2 are equivalent knots, then $\tau_3(K_1) = \tau_3(K_2)$.
 - (c) If an oriented link has $lk(L) = 0$, then L is equivalent to the unlink.
 - (d) There is no knot with $\tau_3(K) = 6$.
 - (e) For any knot $\tau_3(\overline{K}) = -\tau_3(K)$.

- (f) The function f defined on knot diagrams by setting

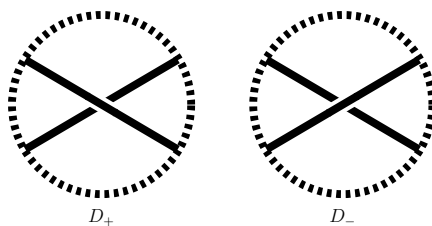
$$f(D) = \text{number of crossings on } D$$

defines an invariant of knots.

- (g) For any n , the number of knots with crossing number at most n is finite.
- (h) For any n , the number of knots with unknotting number at most n is finite.
- (i) For any oriented knot K , $\tau_5(rK) = \tau_5(K)$.
- (j) For any knot $\tau_2(K) = 2$.
10. Show that the equation that needs to be satisfied for 3-colourability is equivalent to the general equation for p -colourability.
11. Show that τ_2 equals 2 to the power of the number of components of the link and is therefore not very interesting.
12. Let τ_5 be the number of 5-colourings of a link.
- (a) Use τ_5 to show that the figure-eight knot 4_1 is non-trivial.
- (b) Calculate $\tau_5(4_1 \# 4_1)$
- (c) Suppose the diagram below allows non-trivial 5-colourings with the colours a, b, c and d emerging at the ends as shown. Show that $a = c$ and $b = d$.



13. Calculate the number of 5- and 7-colourings of the knot 7_1 .
14. Suppose K_+ and K_- are two knots having diagrams D_+ and D_- , which are identical except at one crossing, as shown below.



- (a) Prove that $\tau_p(K_+)$ and $\tau_p(K_-)$ are either equal or one is p times the other.
- (b) Deduce that the unknotting number of a knot that satisfies

$$u(K) \geq \log_p(\tau_p(K)) - 1$$

Due Date: 18.03.2019