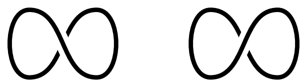
Solutions to sheet 1

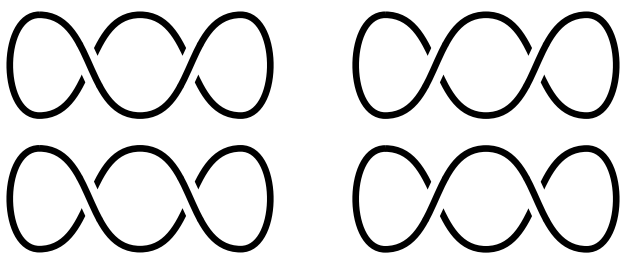
Solution to exercise 1:

We start with showing that there are no knots with crossing number one. For this, note that there are only two ways of completing a single crossing to a knot without introducing any other crossings, namely:



Clearly, these are both diagrams of the unknot.

For crossing number two, there are four possibilities, all of which will again give us the unknot, as shown in the following figure. (One can also check that there are only those four possible ways by drawing two crossings in the plane and go through all possible ways of connecting them without creating more than two crossing and without creating links.)



Hence, there are no knots with crossing number one or two.

Solution to exercise 2:

Begin with drawing three crossings in the plane, without specifying the over-under order yet. The 12 endpoints of our crossings shall be called *ends*. Successively connect two available ends with a strand, such that:

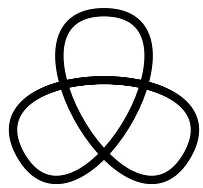
(i) The strand does not introduce any new crossings. (since we want to end up with three crossings)

(ii) The strand does not connect any crossing with itself, since this would create a twist which could be undone, resulting in a lower crossing number, i.e.  is the same as $\supset$.

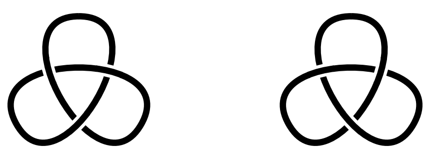
(iii) The strand does not enclose a planar region containing an odd number of available ends, since we would have no way of connecting these ends without creating new crossings (which would violate the first rule), e.g. the following situation is not allowed:

(iv) The strand does not close up a loop, unless the loop contains all 12 ends, since we want to end up with one component only.

The only possible way to connect the three crossings (we did not specify over- and under-crossings yet) looks as follows:

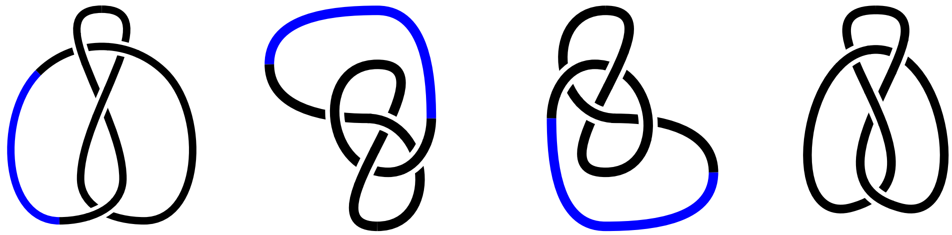


Now, there are exactly 8 ways for choosing the crossings. Fix a point and a direction. When traversing the knot, one can decide at every crossing whether to choose an over- or an under-crossing. This leads to 2*2*2=8 possible knot diagrams with three crossings. By writing down all of these 8 possibilities, we see that except for the trefoil and its mirror image, we end up with the unknot. (It is obvious that the crossings need to be chosen in an alternating fashion, since we create the unknot otherwise.) I.e.



Solution to exercise 3:

We have to show that the figure-eight knot is amphichiral, which means that the knot is the same as its mirror image. This can be achieved by performing the following steps:



The first figure shows the figure-eight knot. We move the lower left part of the knot up, as depicted in the second figure. After that, we rotate the knot by 180° and the in the last step we rearrange everything in a neat way to obtain the mirror image of the figure-eight knot.

Solution to exercise 4:

Remember that we defined the mirror image $\overline{K}$ of a knot K to be a reflection of K in a plane in $\mathbb{R}^3$. So, let us reflect K in the (x-y)-plane. Hence, our reflection map is given by $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (x, y, -z)$. If we now compare two points $P, Q \in K$, we see that for the z-coordinates of those points, denoted by P_z, Q_z, we have: $P_z < Q_z$ if and only if $-Q_z < -P_z$. Looking now at the knot diagram of K and $\overline{K}$ projected to the (x-y)-plane, we see that all over crossings became under crossings and vice versa.

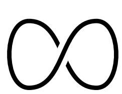
Solution to exercise 5:

Recall that the connected sum $K = K_1 \# K_2$ of two oriented knots is obtained by removing a small segment in each knot and then connecting the open ends of the first knot with those of the second one, according to the orientation. A priori the result of this operation might depend on the position of the segments that we remove. Fortunately this is not the case, as the following consideration shows:

In K we can tighten the part coming from K_1 such that it becomes very small. Then we can move this small piece of K along the part coming from K_2. At any point P we can stop and enlarge the contribution of K_1 again. The result is again the connected sum, this time with the segment of K_2 being removed at P.

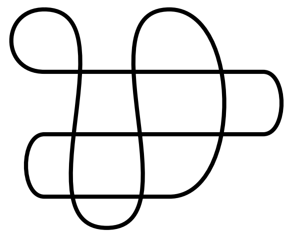
Solution to exercise 6:

A knot is called alternating if an alternating diagram of the knot can be found. Clearly, the following diagram works for the unknot, which proves the claim:

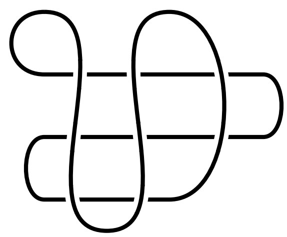


Solution to exercise 7:

We start drawing a random immersion of the circle in the plane, which could look for example like the following figure:



The idea is to choose the crossings as follows: Start at the furthest most left point of the curve and traverse it from there. Each time you arrive for the first time at a crossing you choose an over-crossing, once you arrive for the second time at a crossing you must choose an under-crossing. The result is that you wind down the curve and it is nothing more than a rolled up piece of rope without any knots in it. In the example above, we obtain:



To make this more precise, consider a parameterization of a knot projection (i.e. an immersed curve) given by

$$r : [0,1] \to \mathbb{R}^2;\, r(s) = (x(s), y(s))$$

with $(x(0), y(0)) = (x(1), y(1)) = (x_L, y_L)$ where (x_L, y_L) is the furthest most left point of the curve (check we can always organize the curve to have such a point). Now lift this curve to $\mathbb{R}^3$ in the following way

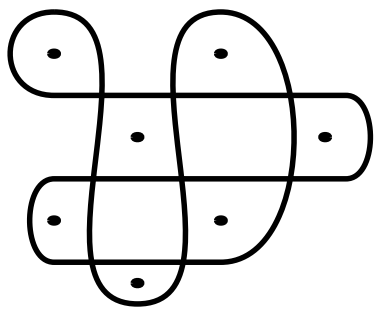
$$(x(s), y(s)) \mapsto (x(s), y(s), 1-s) \quad \text{for} \quad 0 \leq s < 1-\varepsilon$$

and for $1-\varepsilon \leq s \leq 1$ connect the lift with a straight line to its starting point. Check you can always choose ε small enough to guarantee the curve has no self intersection (e.g. $x(1-\varepsilon) < x_L + d$).

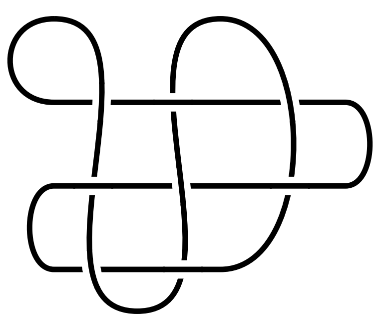
Now consider a different projection of this curve, namely the one onto the $(x-z)$-plane. For $0 \leq s < 1-\varepsilon$ all points of the curve have different z-coordinates (it decreases monotonously) so there are no crossings. By choosing ε small enough we can make sure the straight line connecting to the starting point does not intersect the rest of the curve and therefore, these projections have no crossings at all and hence it must be a projection of the unknot. And hence the knot we have created is the unknot, as we can find a knot diagram with zero crossings.

Solution to exercise 8:

Assume that we have a chessboard coloring of the regions of the knot diagram, which means that each of the regions is colored black or white, in such a way that adjacent regions have different colors. For our example in exercise 7, this gives



where the black dots indicate the areas which are colored in black. Now choose an orientation for the diagram, as for example starting at the left most point and starting upwards. At every crossing exactly one of the two strands has the following property: After passing through the double point (in the direction prescribed by the chosen orientation) the region on the left side of the strand is colored in black. (If both or none of the strands had this property we wouldn't have a proper chess board coloring). We let this strand go over the other one. The resulting diagram is alternating: Following a strand that goes over the other one at some crossing we know that the region on its right is colored in black. Hence, after passing through the subsequent crossing, the region on its left will be the black one (by the chessboard property). Hence the strand will pass under the other one at this second crossing. For our example, we get:



(The problem of finding a chessboard coloring is a slight variation of the first part of problem 5.a of the next exercise sheet. The idea is the following: We fix some point P_0 which is not contained in our knot projection. To determine the color of a region R we pick some point P in it and connect it with P_0 using a straight line segment ℓ. We let $N(R)$ be the number of intersection points of ℓ with the strands of the diagram. If $N(R)$ is even R is colored in white, if $N(R)$ is odd we color it in black.)