Solutions to sheet 10

Solution to exercise 1:

(a) We need to make the problem more precise: We need to assume that X is a combinatorial surface which we can write as a combinatorial union of sub-surfaces $X = A \cup B$. To compute $\chi(X) = \chi(A \cup B)$ we note that

$$\begin{aligned}&\#\{\text{vertices in } X\}\\&= \#\{\text{vertices in } A\} + \#\{\text{vertices in } B\} - \#\{\text{vertices in } A \cap B\}\end{aligned}$$

and the same holds for the respective numbers of edges and triangles, hence

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

(b) A combinatorial circle C is nothing but a polygonal loop. If C has n vertices then C has n edges, hence $\chi(C) = n - n = 0$.

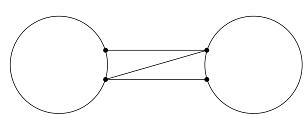
Solution to exercise 2:

(a) From problem 4 of exercise sheet 9 we know that the boundary of F' is the boundary of F plus one or two combinatorial circles. By the previous problem these circles do not contribute to the Euler characteristic.

(b) The Euler characteristic of a combinatorial disc is equal to 1. There are 1 or 2 new boundary components, if we cap them off with discs the Euler characteristic increases by 1 or 2 respectively, which follows from Problem 1a.

Solution to exercise 3:

(a) Start with an arrangement X of d disjoint discs. We have $\chi(X) = d$. Now let X' be the space obtained by adding a combinatorial band (a rectangle subdivided diagonally into two triangles) as in this picture:



We have 2 new triangles and 3 new edges. Hence we have $\chi(X') = \chi(X) - 1$. So if Y is obtained by adding b bands to X then $\chi(Y) = d - b$.

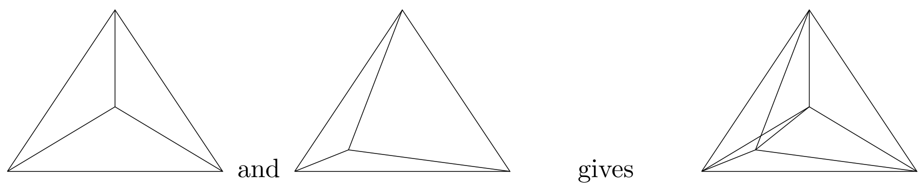
(b) This surface S is non-orientable, it has one boundary component and Euler characteristic -1, since a cylinder has $\chi = 0$ and adding a band decreased χ by 1 as we have seen above. The classification of surfaces tells us that $-1 = \chi(S) = 2 - g(S) - 1$, which implies that the genus $g(S)$ is equal to 2. Hence S is a Klein bottle with a disc removed.

Solution to exercise 4:

A triangulation of the sphere can be obtained, for example, by radially projecting a tetrahedron T whose vertices lie on the sphere. The fact that any two triangulations yield the same Euler characteristic follows from these two facts:

(i) The Euler characteristic of a triangulation does not change when we refine the triangulation:

(ii) Any two triangulations (of the sphere) have a common refinement, e.g.



Solution to exercise 5:

This follows from previous problems: Start with $F_1 \sqcup F_2$ and remove a disc in both surfaces to obtain $F_1' \sqcup F_2'$. Removing a disc decreases χ by 1, so we have $\chi(F_1' \sqcup F_2') = \chi(F_1 \sqcup F_2) - 2 = \chi(F_1) + \chi(F_2) - 2$. Now gluing F_1 with F_2 along the new boundary components doesn't change χ by Problem 2a, hence $\chi(F_1 \# F_2) = \chi(F_1) + \chi(F_2) - 2$. The Euler characteristic and the genus of an *orientable* surface F are linked via $\chi(F) = 2 - 2g(F)$. By what we just proved we have

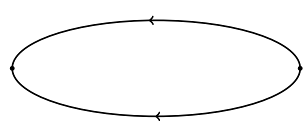
$$\begin{aligned} 2 - 2g(F_1 \# F_2) &= \chi(F_1 \# F_2) \\ &= \chi(F_1) + \chi(F_2) - 2 \\ &= 2 - 2g(F_1) + 2 - 2g(F_2) - 2 \\ &= 2 - 2(g(F_1) + F_2)) \end{aligned}$$

and hence, $g(F_1 \# F_2) = g(F_1) + g(F_2)$.

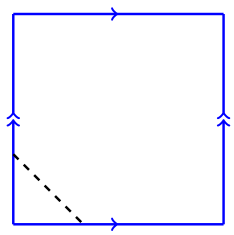
Solution to exercise 6:

(a) This glueing yields a cone, which, as a topological surface, is a disc.

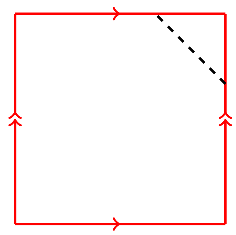
(b) The simplest model is a bigon:



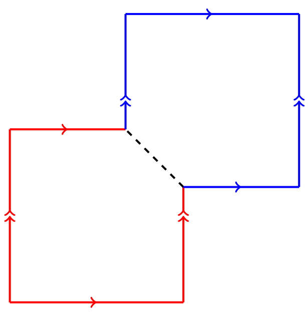
(c) In the following we take two copies of the above planar model for Σ_1. A disc can be removed in the surface by adding an additional edge to the polygon. I.e. we start with



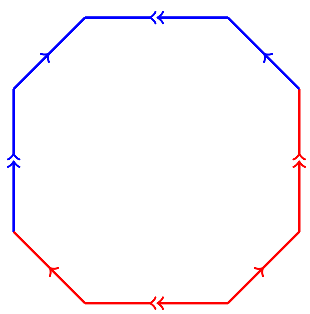
and



Then we glue the two new edges, which means that we glue the surfaces along the boundaries of the removed discs. We get:

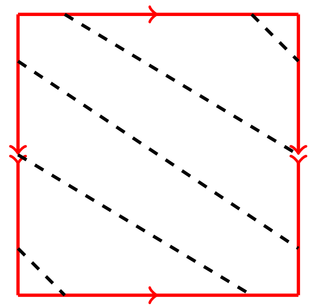


This can also be seen as this octagon:



(d) Convince yourself by cutting it out and glueing the corresponding edges.

(e) The model looks as follows:



Solution to exercise 7:

(a) Proceed as in the previous exercise sheets to deform the boundary curve into a diagram of the trefoil knot.

(b) This is because an untwisted band and a band twisted by 360 degrees are homeomorphic via a homeomorphism that is the identity on the vertical boundary segments. In other words, the twisting of the bands is a feature of the given embedding, not of the surface itself.

(c) An ambient isotopy between the two surfaces (i.e. between the one with twisted bands and the one with untwisted bands) induces an an isotopy of the boundaries. This is impossible, since the surface with untwisted bands has the unknot as its boundary, while the initial surface has a trefoil boundary.

Solution to exercise 8:

As in problem 7b the untwisted band in the left surface is homeomorphic, via a map preserving the boundary, to the knotted and twisted 'pretzel' band in the right surface. A homeomorphism from the left to the right surface will be this map on the untwisted band and the identity everywhere else.

Solution to exercise 9:

These are two diagrams of the trefoil knot. The left one is easily transformed into the right one by taking the left-most string to the right. We have seen in an earlier sheet that the right one is the trefoil knot. We make a chess-board colouring and leave the outside white. Then the left surface is non-orientable, the right surface is orientable.

Solution to exercise 10:

The surface obtained by the chessboard coloring can be seen as a surface made from disks and bands. This allows us to compute the Euler characteristic (using problem 3a). Together with the orientability we can then identify the surfaces. The surface S_1 on the left has 2 disks and 5 bands, i.e. $\chi(S_1) = 2 - 5 = -3$ It is orientable and it has 5 boundary components. We have

$$-3 = \chi(S_1) = 2 - 2g(S_1) - 5$$

and hence, S_1 is the orientable surface of genus 0 with 5 boundary components, i.e. a sphere with 5 disks removed.

The surface S_2 on the right has 3 disks and 8 bands, i.e. $\chi(S_2) = 3-8 = -5$. It is non-orientable and it has 3 boundary components. We have

$$-5 = \chi(S_2) = 2 - g(S_2) - 3$$

and hence, S_2 is the non-orientable surface of genus 4 with 3 boundary components.

Solution to exercise 11:

This surface S is made from 4 disks and 7 bands, i.e. $\chi(S) = 4 - 7 = -3$. It is orientable and it has 3 boundary components. We have

$$-3 = \chi(S) = 2 - 2g(S) - 3$$

and hence, S is the orientable surface of genus 1 with 3 boundary components, i.e. a torus with 3 disks removed.