Solutions to sheet 11

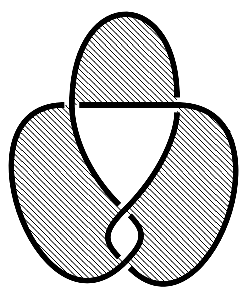
Solution to exercise 1:

Note that when changing the orientation of a knot, every positive crossing stays a positive crossing and every negative crossing stays a negative crossing. Hence, both orientations yield the same Seifert circles but different orientations for those. Since the bands between the circles are independent of the orientation, the whole surface does not depend on the chosen orientation.

Solution to exercise 2:

(a) Using Seifert's algorithm to construct a Seifert surface F_1 of the trefoil, we obtain 2 discs and 3 bands for the trefoil. Hence $\chi(F_1) = -1$ and $g(F_1) = 1$. Since Seifert's algorithm always yields an orientable surface, this is a torus with one boundary component. Recall that in the lecture, using the chessboard coloring of the trefoil, we obtained a non-orientable surface, namely a punctured sphere with a crosscap attached.

(b) A chessboard coloring of the figure-eight knot yields the surface F_2, given by:



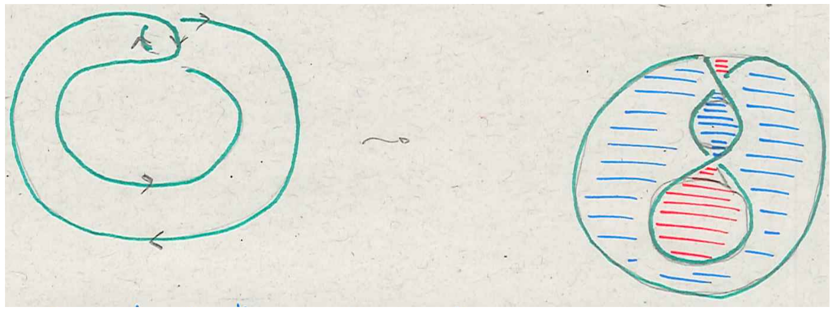
This is a non-orientable surface with 3 discs and 4 bands, i.e. we have $\chi(F_2) = -1$ and $g(F_2) = 2$. So, we get a punctured Klein bottle. Recall that Seifert's algorithm gives us another surface, since we necessarily obtain an oriented surface. We have seen in class that Seifert's algorithm yields a torus with one boundary component.

Solution to exercise 3:

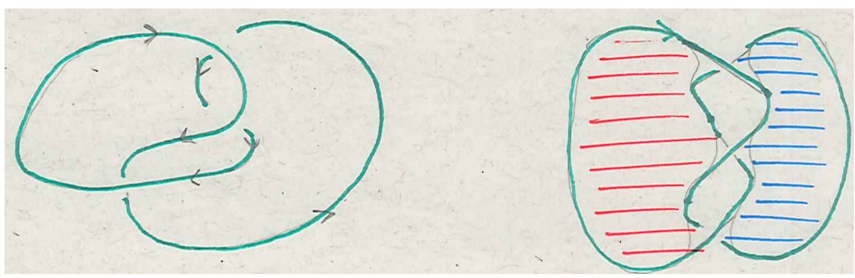
(a) The disc has Euler characteristic $\chi = 1$, is orientable and has one boundary component, hence it has genus $g = \frac{1}{2}(2 - 1 - 1) = 0$.

(b) Seifert's algorithm produces two discs joined by a twisted band, thus (see Exercise 3 on Sheet 9) $\chi = 2 - 1 = 1$ and again for the genus $g = 0$.

(c) Seifert's algorithm produces three discs joined by two twisted bands, thus $\chi = 3 - 2 = 1$ and $g = 0$.



(d) Seifert's algorithm gives a surface with two discs joined by three bands, hence $\chi = 2 - 3 = -1$ and the genus is $g = \frac{1}{2}(2 - 1 + 1) = 1$. Therefore it's a torus minus a disc.

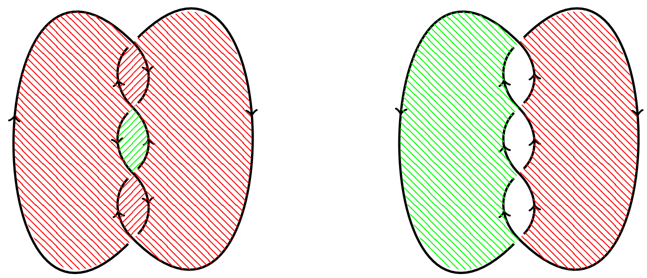


Solution to exercise 4:

This is a link with two components. Hence any surface F we will obtain using Seifert's algorithm has two boundary components. We can use the relation

$$g(F) = \frac{2 - \chi(F) - 2}{2}.$$

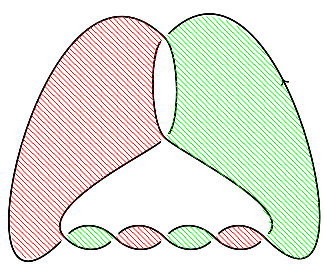
The Seifert surface F_1 we obtain by orienting both strings clockwise has 4 discs and 4 bands, which gives $\chi(F_1) = 0$ and hence $g(F_1) = 0$. The Seifert surface F_2 we obtain by orienting one string clockwise and the other string counter clockwise has 2 discs and 4 bands, which gives $\chi(F_1) = -2$ and hence $g(F_2) = 1$.



As the genus of the link should be the minimal genus over all Seifert surfaces, we obtain $g(L) = 0$.

Solution to exercise 5:

Look at the Seifert surface constructed from the first knot give:



There are 6 discs and 7 bands. Hence, we have $\chi(S_1) = -1$ and $g(S_1) = 1$, i.e./ we obtain a punctured torus.

Similarly can be proceeded for the other two knots. We obtain $\chi(S_2) = -5$ and $g(S_2) = 3$ for the second knot and $\chi(S_3) = -3$ and $g(S_3) = 2$ for the third knot.

Solution to exercise 6:

Note that this diagram has 8 crossings, the bands passing one above the other produce 4 crossings. Using the notation from the previous problem we obtain $d = 5$, $b = 8$, $\chi = -3$, $g = 2$.

Solution to exercise 7:

In our previous notation we have $s = d$ (Seifert circles are discs) and $c = b$ (crossings correspond to bands). For the Seifert surface we have the relations $\chi = 2-2g-1 = 1-2g$ (here we're using that there is one boundary component) and $\chi = s - c$ (from the previous problem sheet). This yields the formula

$$g = \frac{1}{2}(1 - s + c).$$

Solution to exercise 8:

Since we obtain always at least one Seifert circle (disc) using the Seifert algorithm we have $1 - s \leq 0$, so the formula of the previous problem yields $g \leq \frac{1}{2}c$.

Write $g(D)$ and $c(D)$ for the genus of the Seifert surface obtained from a knot diagram D, and $c(D)$ for the crossing number of D. We just noted that $g(D) \leq \frac{1}{2}c(D)$. Let $D_{\min}$ be a diagram with minimal crossing number for a knot K. We have

$$g(K) \leq g(D_{\min}) \leq \frac{1}{2}\,c(D_{\min}) = \frac{1}{2}\,c(K).$$

Solution to exercise 9:

Let K have genus 1, and let $K = K_1 \# K_2$. We have

$$1 = g(K) = g(K_1) + g(K_2)$$

Since $g \geq 0$, one of K_1 and K_2 has genus 0 and is therefore the unknot.

Solution to exercise 10:

(a) $g(6_3) = 2$, $g(7_6) = 2$

(b) The converse doesn't hold, by (a) there are prime knots which have genus greater than 1.

Solution to exercise 11:

(a) The Seifert Surface S has $d = 3$, $b = 6$, $\chi = -3$, $g = 2$. Hence, S is the orientable surface of genus 2 with one boundary component.

(b) The given diagram can obviously be simplified with a Reidemeister 2 move. The result is the standard diagram of the figure eight knot, which we know is prime. Furthermore, the new diagram is alternating, hence Seifert's algorithm yields a Seifert surface S of minimal genus. One finds that S has $d = 3$, $b = 4$, $\chi = -1$ and $g = 1$.

Remark: Note that the genus of the Seifert surface depends on the chosen diagram. In particular, we have seen in this example that performing a Reidemeister 2 move can change the genus of the Seifert surface. That is because the number of bands is increased by 2 when performing a Reidemeister 2 move, but the number of discs stays the same. Also note that a Reidemeister 1 move does not change the genus, since we add one band as well as one disc and henceforth, the Euler characteristic stays the same.

Solution to exercise 12:

Let K_1 be any non-trivial knot, e.g. the trefoil knot. The infinite sequence $(K_n)_{n\geq 1}$ given by $K_n = K_1^{\# n}$ (n-fold connected sum) consists of pairwise inequivalent knots as the genera $g(K_n) = n \cdot g(K_1)$ are pairwise distinct.

Solution to exercise 13:

Let U be the unknot. If $K \# J = U$ then $0 \leq g(K) + g(J) = g(U) = 0$, i.e. $g(K) = g(J) = 0$, i.e. $K = J = U$. This means that only the unknot has an inverse.