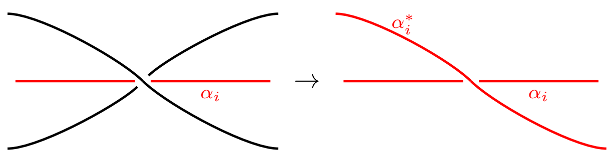
Solutions to sheet 12

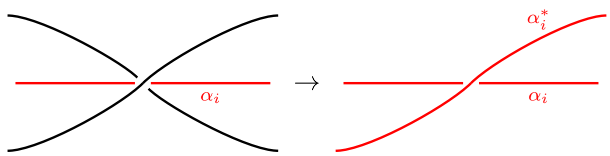
Solution to exercise 1:

We make two general observations that allow an efficient computation of the Seifert matrix. Let $\alpha_1, \ldots, \alpha_{2g}$ be the curves on a Seifert surface S. By α_i^* we denote the curve α_i moved a little bit into the positive normal direction, which is naturally given by the orientation of K (and hence by the orientation of the surface S).

(i) Let the curve α_i go along a band. Then a twist in this band contributes $+\frac{1}{2}$ or $-\frac{1}{2}$ to $\mathrm{lk}(\alpha_i, \alpha_i^*)$. More precisely, it contributes $+\frac{1}{2}$ for

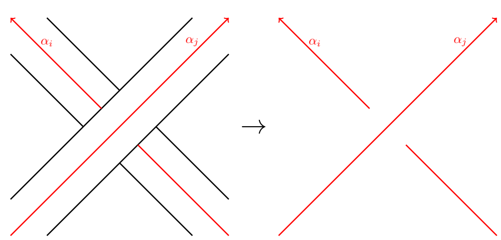


and $-\frac{1}{2}$ for



Note that the orientation of α_i^* is determined by the orientation of α_i and does not play a role. Note also that for the self-linking, only the direction of the twist in a band (i.e. over- or under-crossing) plays a role, but not the orientation of the knot K.

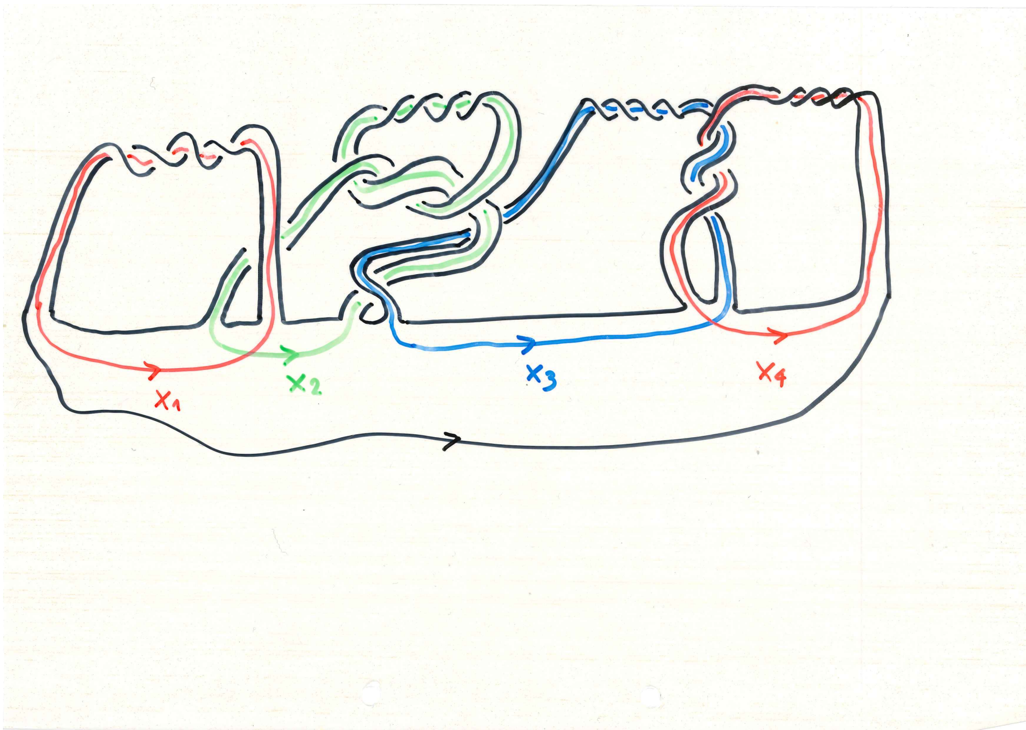
(ii) If α_i and α_j go along bands and one of the bands passes over the other then this contributes $+\frac{1}{2}$ or $-\frac{1}{2}$ to $\mathrm{lk}(\alpha_i, \alpha_j^*)$ as well. In the following situation for example, it contributes $+\frac{1}{2}$:



Note that we simply treat the bands as if they were strands. Also note that $i = j$ is allowed here, i.e. the observation applies when the band for α_i passes over itself.

Assuming that the α_i are chosen in such a way that no band is passed by two distinct curves, we are left with considering crossings of curves that happen inside the discs of the Seifert surface.

Now let's apply these observations to the example from class:



We have

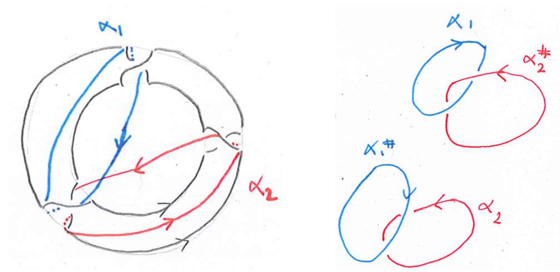
- $\mathrm{lk}(x_1, x_1^*) = 2$ since there are 4 twists in the band for x_1 each contributing $+\frac{1}{2}$.
- $\mathrm{lk}(x_3, x_4^*) = -1$ since there are 3 over-passes between the bands for x_3 and x_4, each contributing $-\frac{1}{2}$ and one crossing in the disc contributing $+\frac{1}{2}$.
- $\mathrm{lk}(x_4, x_3^*) = -2$ since there are 3 over-passes between the bands for x_4 and x_3, each contributing $-\frac{1}{2}$ and one crossing in the disc contributing $-\frac{1}{2}$.
- etc.

The resulting Seifert matrix is

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -2 & -2 \end{pmatrix}$$

Solution to exercise 2:

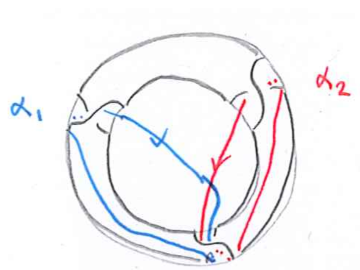
(a) Consider first the left-hand trefoil and the Seifert surface given by Seifert's algorithm. It has genus 1 so we have to put 2 curves on it.



Since both α_1 and α_2 pass through 2 twists each, each contributing $\frac{1}{2}$, it holds that $\operatorname{lk}(\alpha_1, \alpha_1^{\#}) = 1$ and $\operatorname{lk}(\alpha_2, \alpha_2^{\#}) = 1$. Moreover $\operatorname{lk}(\alpha_1, \alpha_2^{\#}) = 1$ and $\operatorname{lk}(\alpha_2, \alpha_1^{\#}) = 0$, so the Seifert's matrix of the left-hand trefoil is given by

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

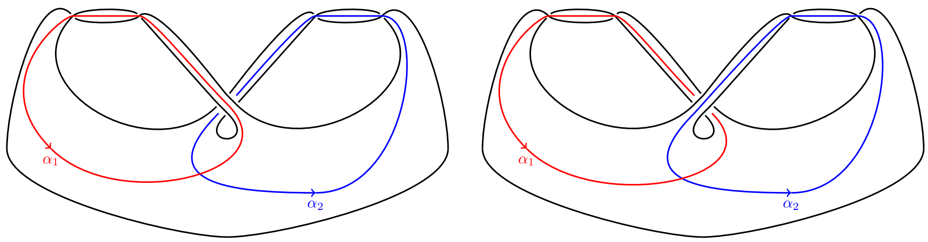
Similarly for the right-hand trefoil we get



Hence $\operatorname{lk}(\alpha_1, \alpha_1^{\#}) = -1$, $\operatorname{lk}(\alpha_2, \alpha_2^{\#}) = -1$, $\operatorname{lk}(\alpha_1, \alpha_2^{\#}) = 0$ and $\operatorname{lk}(\alpha_2, \alpha_1^{\#}) = -1$. Thus

$$R = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$$

(b) These surfaces have genus 1, so we have to put 2 curves on each:



We note the following:

- $\mathrm{lk}(\alpha_1, \alpha_1^*) = -1$ since the band for α_1 has two twists contributing $-\frac{1}{2}$ each.
- $\mathrm{lk}(\alpha_1, \alpha_2^*) = 1$ since there is one over-pass between the bands for α_1 and α_2 which contributes $+\frac{1}{2}$ and one crossing in the disc which contributes $+\frac{1}{2}$.
- $\mathrm{lk}(\alpha_2, \alpha_1^*) = 0$ since there is one over-pass between the bands for α_2 and α_1 which contributes $+\frac{1}{2}$ and one crossing in the disc which contributes $-\frac{1}{2}$.
- $\mathrm{lk}(\alpha_2, \alpha_2^*) = -1$ since the band for α_2 has two twists contributing $-\frac{1}{2}$ each.

Hence the Seifert matrix for the surface on the left is

$$L = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

In the same way we compute the Seifert matrix for the second surface. The result is

$$R = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

(c) In our example we have the relation $R = -L^T$. More generally, if M_D is the Seifert matrix obtained from a diagram D then $M_{\overline{D}} = -M_D^T$.

To see this let $\{\alpha_i\}$ be a chosen collection of curves on the Seifert surface for D and let $\{\beta_i\}$, $\beta_i = \overline{\alpha_i}$ be the corresponding collection on the surface for the mirror image. Denote by $\alpha^\#$ the curve α pushed off the surface in negative direction ("away from us"). We have

$$\mathrm{lk}(\alpha_i, \alpha_j^*) = \mathrm{lk}(\alpha_i^\#, \alpha_j) = -\mathrm{lk}(\overline{\alpha_i^\#}, \overline{\alpha_j}) = -\mathrm{lk}(\beta_i^*, \beta_j) = -\mathrm{lk}(\beta_j, \beta_i^*)$$

which proves the claim.

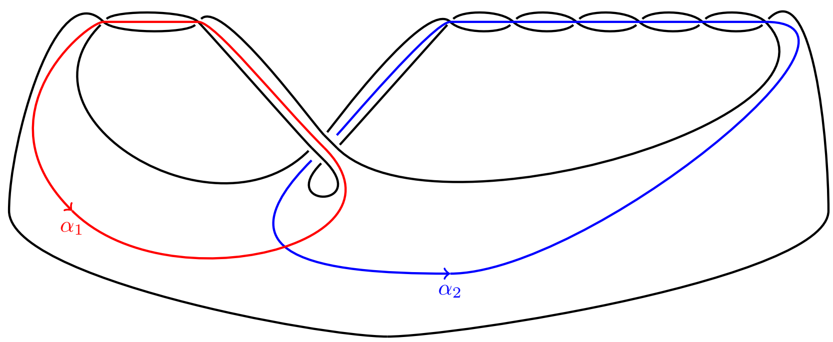
In particular, if M is any Seifert matrix for K and N is any Seifert matrix for $\overline{K}$ then $M \overset{S}{\sim} -N^T$.

Note that the first surface of (b) is a Seifert matrix for the right-hand trefoil and in fact it holds that

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$$

Solution to exercise 3:

This is again a surface of genus 1. As in the previous problem, we orient the two curves that go along the bands in counter-clockwise direction.



The resulting Seifert matrix is

$$\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}$$

If the orientation or the order of the two curves is changed the resulting Seifert matrix will be different from the given one by a Λ_1 operation.

Solution to exercise 4:

Let α be a curve on a Seifert surface S. Recall that we denote by α^* and $\alpha^\#$ the push-offs of this curve in the two possible directions. These directions correspond to orientations of the surface. Let $\alpha_1, \ldots, \alpha_{2g}$ be a chosen collection of curves on S. The Seifert matrices for the two orientations of S are $M = (\mathrm{lk}(\alpha_i, \alpha_j^*))_{ij}$ and $N = (\mathrm{lk}(\alpha_i, \alpha_j^\#))_{ij}$. We have

$$\mathrm{lk}(\alpha_i, \alpha_j^\#) = \mathrm{lk}(\alpha_i^*, \alpha_j) = \mathrm{lk}(\alpha_j, \alpha_i^*).$$

Hence we have the relation $N = M^T$. Note that M and N are *not* S-equivalent, in general. The push-off direction on S must be chosen such that it is consistent (right-hand rule) with the orientation of the boundary curve (= the knot).

Solution to exercise 5:

Let M be the original Seifert matrix. The given manipulations induce the Seifert equivalence $M' = PMP^T$, where P is the matrix:

(a)

$$\begin{array}{c} \qquad\qquad\qquad i \\ \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ i \\ \\ \\ \\ \end{matrix} \end{array}$$

(b)

$$\begin{array}{c} \qquad\quad i \qquad\qquad\qquad j \\ \begin{pmatrix} 1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & 1 & & & & & & & \\ & & & 0 & & & & 1 & & \\ & & & & 1 & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & 1 & & & \\ & & & 1 & & & & 0 & & \\ & & & & & & & & 1 & \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ i \\ \\ \\ \\ j \\ \\ \\ \\ \end{matrix} \end{array}$$

(c)

$$\begin{array}{c} \qquad\qquad\qquad j \\ \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & \ddots & & 1 & & \\ & & & \ddots & & & \\ & & & & \ddots & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \begin{matrix} \\ \\ i \\ \\ \\ \\ \\ \end{matrix} \end{array}$$

Solution to exercise 6:

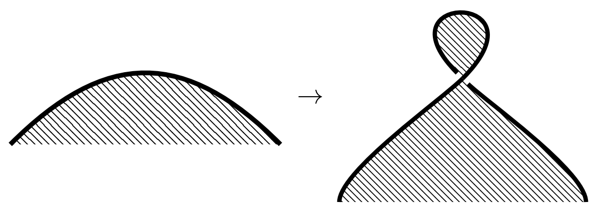
Let M_1, M_2 be the Seifert matrices for F_1 and F_2. The Seifert matrix for $F_1 \# F_2$ is the block matrix

$$\left(\begin{array}{c|c} M_1 & 0 \\ \hline 0 & M_2 \end{array}\right)$$

This is because the curves on M_1 and those on M_2 are pairwise separated, hence unlinked, in the connected sum.

Solution to exercise 7:

Performing a Reidemeister 1 move means that we have added one disc and one band in the resulting Seifert surface.



Since $\chi = d - b$ the Euler characteristic is unchanged, and so is the genus. Therefore there's no need for new curves and so the Seifert matrix is unchanged as well. Note that geometrically you add a disc to the surface, but that does not change its topology.

Solution to exercise 8:

(a) If S is a Seifert surface for a diagram D of K then the surface S' which is S with opposite orientation is a Seifert surface for the reverse rD. We have seen in problem 4 that the corresponding Seifert matrices satisfy $M' = M^T$. This implies the claim.

(b) We obtain a diagram $\overline{D}$ for $\overline{K}$ from K by changing the under- and over-crossing segments at each of the crossing points. Therefore, since the under and over relations for the closed curves that follow from D and $\overline{D}$ are completely reversed we have that $M_{\overline{K}}$ is Seifert equivalent to $-M_K^T$.

Solution to exercise 9:

Let M be a Seifert matrix with non-zero determinant, e.g. a matrix from problem 2a. The operation $M \to \Lambda_2(M)$ (addition of a handle) produces a matrix with determinant 0, since its last line (or last column) is zero. Hence the determinant of the Seifert matrix of a knot is not well-defined.

Solution to exercise 10:

(a) Let M be a Seifert matrix. Consider first the transformation $\Lambda_1(M) = PMP^T$. We have

$$\begin{aligned}|\det(\Lambda_1(M)+\Lambda_1(M)^T)| &= |\det(PMP^T + PM^TP^T)|\\ &= |\det(P)\det(M+M^T)\det(P^T)|\\ &= |\det(M+M^T)|,\end{aligned}$$

where we have used standard properties of the determinant and $|\det(P)| = 1$. In case of the second transformation Λ_2 we have to understand the determinant of $\Lambda_2(M)+\Lambda_2(M)^T$. For both variants of the transformation this matrix has the shape

$$\left(\begin{array}{ccc|cc} & & & * & 0 \\ M & + & M^T & \vdots & 0 \\ & & & * & \\ \hline * & \dots & * & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \end{array}\right)$$

We use the following Proposition: If

$$T = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

is a block matrix with A and D square and D invertible, then

$$\det(T) = \det(A - BD^{-1}C)\det(D).$$

This applies to our matrix above, which comes with the indicated block structure. A computation shows that $BD^{-1}C = 0$ and $\det(D) = -1$, so

$$\begin{aligned}|\det(\Lambda_2(M)+\Lambda_2(M)^T)| &= |\det(A)\det(D)|\\ &= |-\det(A)|\\ &= |\det(A)|\\ &= |\det(M+M^T)|.\end{aligned}$$

(b) The unknot U has determinant 1, which, by definition, is the determinant of the empty matrix. If you're uncomfortable with this argument you can compute the determinant of U using a Seifert matrix corresponding to a genus 1 Seifert surface for U, i.e. a torus with a disc removed. To compute the determinant of the trefoil knot we can use the Seifert matrix L from problem 2(a), we have

$$L + L^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and hence $|\det(L + L^T)| = 3$, which shows that the trefoil knot is non-trivial.

(c) It turns out, after a not-so-short calculation, that this knot has determinant 1. Using other invariants, e.g. the Jones polnomial, one can show that the given knot is non-trivial. In fact, the given knot is the prime knot 10_{124}. This means that the determinant cannot detect the unknot.