Solutions to sheet 13

Solution to exercise 1:

We obtain

$$24t^{-2} - 188t^{-1} + 337 - 188t + 24t^2.$$

Solution to exercise 2:

We obtain

$$3t^{-1} - 5 + 3t.$$

Solution to exercise 3:

(a) By Exercise 8(a) of sheet 12, we have $M_{rK} = M_K^T$, so

$$\begin{aligned}\Delta_{rK}(t) &= t^{-k/2}\det(M_K^T - tM_K)\\ &= t^{-k/2}\det(M_K - tM_K^T)\\ &= \Delta_K(t).\end{aligned}$$

Here we have used that $\det(X^T) = \det(X)$.

(b) By Exercise 8(b) of sheet 12 we have $M_{\overline{K}} = -M_K$, so

$$\begin{aligned}\Delta_{\overline{K}}(t) &= t^{k/2}\det(-(M_K - tM_K^T))\\ &= (-1)^k \cdot t^{-k/2}\det(M_K - tM_K^T)\\ &= t^{-k/2}\det(M_K - tM_K^T)\\ &= \Delta_K(t).\end{aligned}$$

Here we have used that k is twice the genus and hence even.

Solution to exercise 4:

The formula is

$$\Delta_{K_1\#K_2}(t) = \Delta_{K_1}(t)\cdot\Delta_{K_2}(t).$$

Proof: Let M_1, M_2 be the respective Seifert matrices. Our result in problem 6 of sheet 12 says that

$$\begin{aligned}\Delta_{K_1\#K_2}(t) &= t^{-(k_1+k_2)}\det\left(\begin{array}{c|c} M_1 - tM_1^T & 0\\ \hline 0 & M_2 - tM_2^T\end{array}\right)\\ &= t^{-k_1/2}\det(M_1 - tM_1^T)\cdot t^{-k_2/2}\det(M_2 - tM_2^T)\\ &= \Delta_{K_1}(t)\cdot\Delta_{K_2}(t).\end{aligned}$$

Solution to exercise 5:

Let S be a Seifert surface with minimal genus $g(K)$ for K, and let M be a corresponding Seifert matrix. The determinant of the $(2g\times 2g)$-matrix $M-tM^T$ is a polynomial $P \in \mathbb{Z}[t]$, with degree at most $2g$. (This follows easily from the definiton of the determinant). Hence we have that $\Delta_K(t) = t^{-g}P(t)$ has degree at most g.

Solution to exercise 6:

We obtain

$$\Delta_{4_1}(t) = -t^{-1} + 3 - t.$$

Solution to exercise 7:

Consider any diagram for K. We look at an unknotting sequence for the diagram. Let K_+ be the diagram before the first step and K_- the diagram after (or vice versa if there is no positive crossing to begin with). The skein relation evaluated at $t = 1$ reads

$$\Delta_{K_+}(1) = \Delta_{K_-}(1)$$

By iteratively using this equation according to the unknotting sequence we find that $\Delta_K(1) = \Delta_{K_+}(1) = \cdots = \Delta_U(1) = 1$.