Solutions to sheet 2

Solution to exercise 1:

Let D be the diagram. From the last problem sheet we know that there is some way of obtaining a diagram U for the unknot by changing crossings of D (Remember that we choose the crossings in such a way that when starting at the leftmost point and traversing the knot in some direction, whenever we come to a crossing the first time, we choose the over-crossing). Say that we obtain U from D by changing k of the n crossings. Observe that we can obtain the mirror image diagram $\overline{U}$, which also represents the unknot, by changing $n-k$ of the crossings of D. The smaller number of k and $n-k$ is at most $\lfloor \frac{n}{2} \rfloor$, and more precisely, it is in particular $\frac{n-1}{2}$ for n odd.

So let us now assume that we have an even number of crossings $n = 2k$ and that the argument used in the last problem sheet yields a diagram of the unknot after changing k of these crossings. We have to refine the argument in order to achieve the sharper upper bound. First of all note that there is no need to start at the leftmost point in the diagram, we can start at any point which is not a crossing. Choose a starting point such that the first crossing C that we pass is an under-crossing. We have to turn it into an over-crossing. In particular, C is one of the k crossings that need to be changed. Now choose a starting point which lies past C. If we run through the diagram starting from this new point we will only have to change the $k-1$ other crossings, since the under strand of C is now reached after the over strand. Thus we have established the bound $\lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{2k-1}{2} \rfloor = k-1$ also in this case.

Solution to exercise 2:

Remember that two embeddings $f_1, f_2 : \mathbb{S}^1 \to \mathbb{R}^3$ are called homotopic, if there exists a continuous function $F : \mathbb{S}^1 \times [0,1] \to \mathbb{R}^3$ such that $F(x,0) = f_1(x)$ and $F(x,1) = f_2(x)$. For this exercise, take the homotopy $F(x,t) := (1-t)f_1(x) + tf_2(x)$, which is clearly continuous. Note: If the knots defined by f_1, f_2 are inequivalent there may be a time parameter t_0 for which the map $S_1 \to \mathbb{R}^3$, $x \mapsto F(x,t_0)$, is not an embedding.

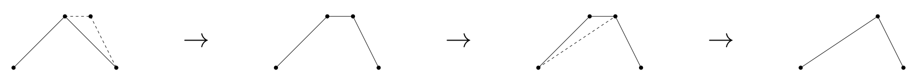
Solution to exercise 3:

We have to show that $K \sim^{\Delta} K'$ is an equivalence relation.

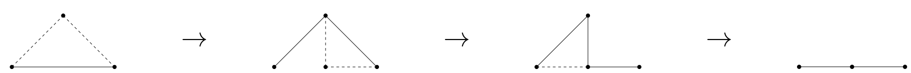
- It is reflexive by considering a sequence of Δ- and Δ^{-1}-moves of length 0.
- Let $K \sim^{\Delta} K'$, i.e. there is a finite sequence of Δ- and Δ^{-1}-moves which takes K into K'. Reversing this sequence takes K' into K, i.e. $K' \sim^{\Delta} K$
- Let $K \sim^{\Delta} K'$ and $K' \sim^{\Delta} K''$, so there is a finite sequence of Δ- and Δ^{-1}-moves which takes K into K' and a finite sequence of Δ- and Δ^{-1}-moves which takes K' into K''. Composition of those sequences yields a finite sequence of Δ- and Δ^{-1}-moves which takes K into K''.

Solution to exercise 4:

Moving one vertex one a knot can be realized by the following two Δ-moves:



Adding a vertex in the middle of any edge can be done by the following three Δ- and Δ^{-1}-moves:



Solution to exercise 5:

Let K be a polygonal knot with vertices $v_1, ..., v_n$. For every vertex v_i, we can find a radius $\varepsilon_i > 0$, such that for all $\tilde{v}_i \in B_{\varepsilon_i}(v_i)$, i.e. the ball around v_i with radius ε_i, moving v_i to $\tilde{v}_i$ (e.g. by the Δ-move seen in exercise 4) does not change the knot. Now define $\varepsilon := \min_{i=1,\dots,n} \varepsilon_i$. Then every such ε-perturbation K' of K is equivalent to K.

Solution to exercise 6:

(a) We begin with choosing a basepoint $x_0 \in C := \mathbb{R}^2 \backslash P$, such that x_0 does not lie on the extension of any of the linear segments of P. We define $N : C \to \mathbb{N}_0$ by

$$N(x) = \#\{p \in [x_0, x] \cap P \,|\, p \text{ is a transversal intersection point}\}$$

With our choice of x_0 every intersection point is transversal unless it is a vertex v of P such that both segments adjacent to v are contained on the same side of the ray $[x_0, x]$. We let $L : C \to \{0, 1\}$ with

$$L(x) = N(x) \mod 2.$$

We claim that L is continuous. Since the target space $\{0, 1\}$ is discrete, this amounts to saying that L is locally constant. I.e. we need to show that for every $x \in C$ there is $\varepsilon > 0$ such that L is constant on $B_\varepsilon(x) \subset C$. So let $x \in C$ be given. Since C is open we can choose $\varepsilon > 0$ such that $B := B_\varepsilon(x) \subset C$. Let $x' \in B$. Consider the ray $R(y) := [x_0, y]$ as y is moving in a straight way from x to x'. The number of intersection points of this moving ray can only change when it sweeps over a vertex v. So let $R(y_0)$ be a ray containing a vertex v. Let $R(y_-)$ and $R(y_+)$ be rays that we see shortly before and shortly after passing v. There are three possible situations:

(i) $R(y_-)$ intersects none of the segments adjacent to v, $R(y_+)$ intersects both of them.

(ii) $R(y_-)$ intersects both segments adjacent to v, $R(y_+)$ intersects none of them.

(iii) $R(y_-)$ intersects one of the segments adjacent to v, $R(y_+)$ intersects the other one.

In the first case $R(y_0)$ intersects P non-transversally at v and we have $N(y_+) = N(y_0) = N(y_-) + 2$. In the second case the intersection is also non-transversal and we have $N(y_+) = N(y_0) - 2 = N(y_-) - 2$. In the third case we have a transversal intersection and $N(y_+) = N(y_0) = N(y_-)$. This means that $L(y)$ is constant as we move y from x to x'. In particular $L(x') = L(x)$, so L is constant on the ball B. Now the map L is clearly surjective and we have a partition $C = C_0 \sqcup C_1$ of C into non-empty open sets $C_0 = L^{-1}(\{0\})$ and $C_1 = L^{-1}(\{1\})$. We have to show that these sets are connected. Let $x, x' \in C_0$. It is sufficient to show that there is a path in C_0 which connects x with x'. To obtain such a path we first connect the two points by a straight line segment $\ell \subset \mathbb{R}^2$. If this segment is contained in C_0 then we are done. If not then there must be points on ℓ which intersect the polygon P. Let $y, y' \in \ell \cap P$ the points in this intersection which are closest to x and x' respectively. Now we have the following path in $\mathbb{R}^2$: We first go from x to y along ℓ, then we proceed along the curve P until we arrive at y' and then we go from y' to x' along ℓ again. Now we take the part of this path which lies on P and slightly shift it off to the C_0-side of P. We then have a path from x to x' within C_0. The same argument shows that C_1 is connected, so C has exactly 2 components.

It remains to show that exactly one of the components is unbounded. This follows from the general fact that the complement of a compact set $K \subset \mathbb{R}^n$ has precisely one unbounded component.

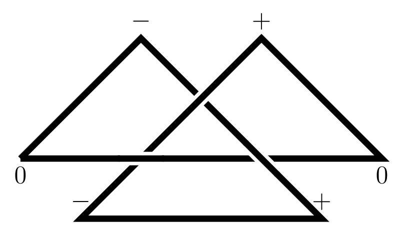
(b) Let $C_0 \subset \mathbb{R}^2$ be the bounded component of C. We subdivide C_0 by cutting it along all lines that are extensions of the linear segments of the polygon P. In the resulting partition $C_0 = A_1 \cup \cdots \cup A_n$ every set A_i is a convex polygonal region. Indeed, if some A_i were not convex then it could be subdivided further (by extending the two edges of a vertex with interior angle larger than $180°$), but we have already done all the possible subdivisions. Now the statement follows from the following observations:

(i) The closure of a convex polygonal region in $\mathbb{R}^2$ is homeomorphic to a closed disc. (Subdivide into triangles, use induction on the number of triangles)

(ii) Gluing two polygonal regions, which are both homeomorphic to a disc, along a common edge yields another region which is homeomorphic to a disc.

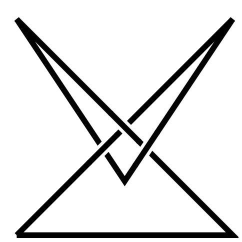
Solution to exercise 7:

(a) The following polygonal diagram represents the trefoil knot. To see that it is indeed a projection of a polygonal knot in $\mathbb{R}^3$ we can think of it as a projection on the plane $\{z = 0\}$, with the vertices having positive/negative/zero z-coordinates as indicated.

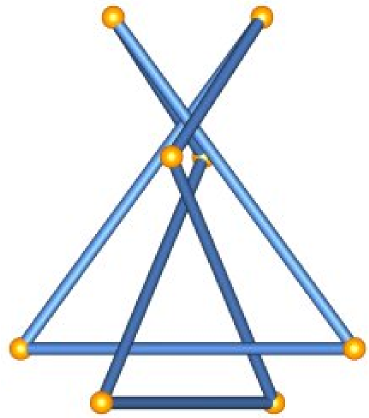


If a polygonal knot has 3 vertices then we have a triangle which clearly represents the unknot. In case of a polygon with 4 vertices note that in a projection, every side of the polygon can cross at most one other side, namely the one to which it is not adjacent. This means that there are at most $4/2 = 2$ crossings in any projection. Hence such a polygon must represent the unknot. To deal with a polygon with 5 vertices we choose a projection in a direction which is parallel to one of the sides. This results in a polygonal knot diagram with 4 vertices, which again has at most 2 crossings.

Note: The following non-trivial polygonal knot diagram has only 5 vertices, but it does not come from a projection of a polygon in $\mathbb{R}^3$.



(b) The following diagram representing 5_1 has 8 vertices.
Source: `www.knotplot.com/cat/ms-symm.html`



Solution to exercise 8:

(a) Take a square in $\mathbb{R}^3$. Any non-cyclic permutation of the vertices will give a polygon with intersecting diagonals.

(b) The vertices in the diagram from the previous exercise can be permuted to obtain a hexagon, which of course represents the unknot.