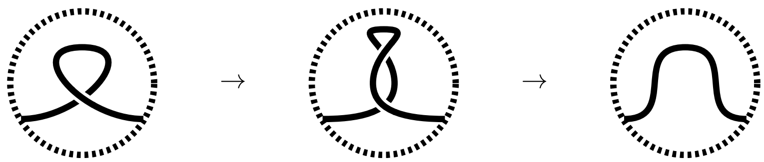
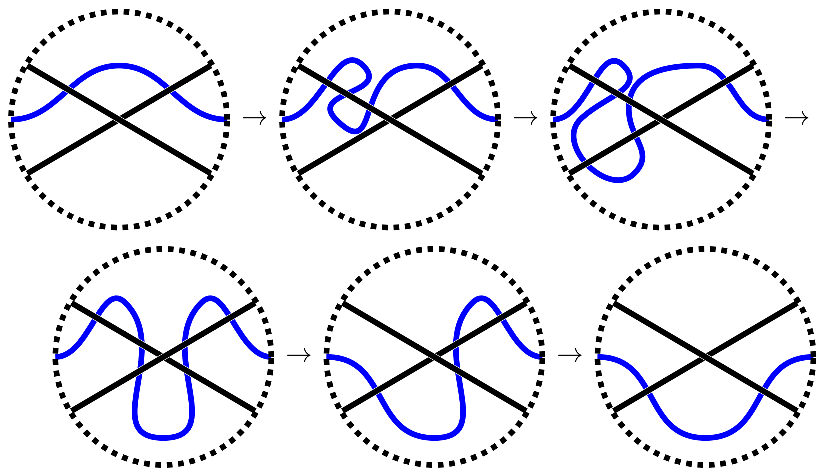
Solutions to sheet 3

Solution to exercise 1:

For the mirror image of $R1$, denoted by $\tilde{R}1$, we can use $R1$ follwed by $R2$, i.e.

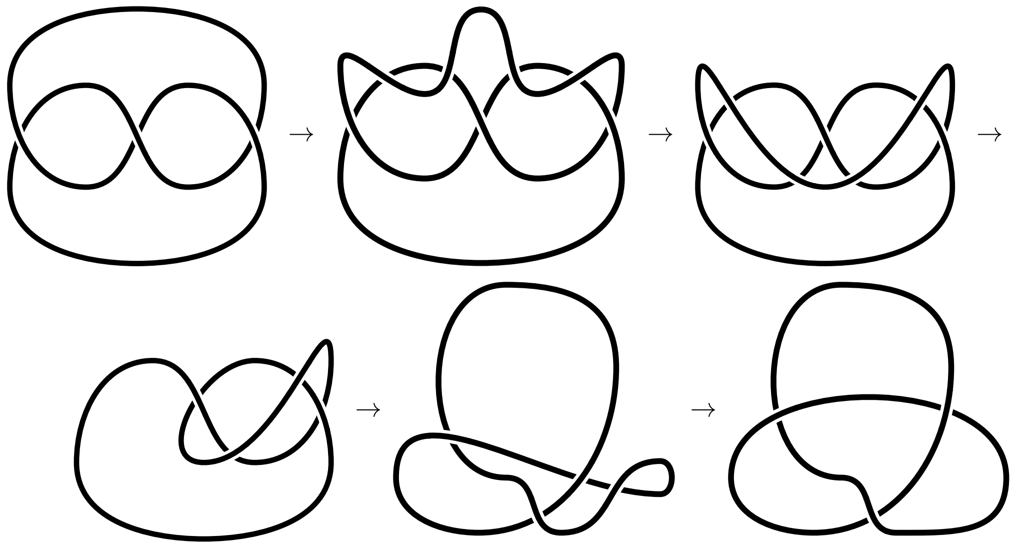


For the mirror image of $R3$, denoted by $\tilde{R}3$, we can use two times $R2$, followed by $R3$, followed by two times $R2$ again.



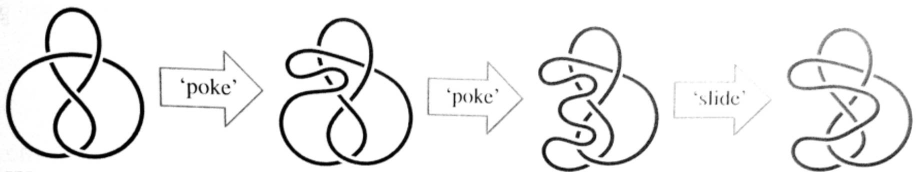
Solution to exercise 2:

For exercise 2a, let's look at the following sequence:

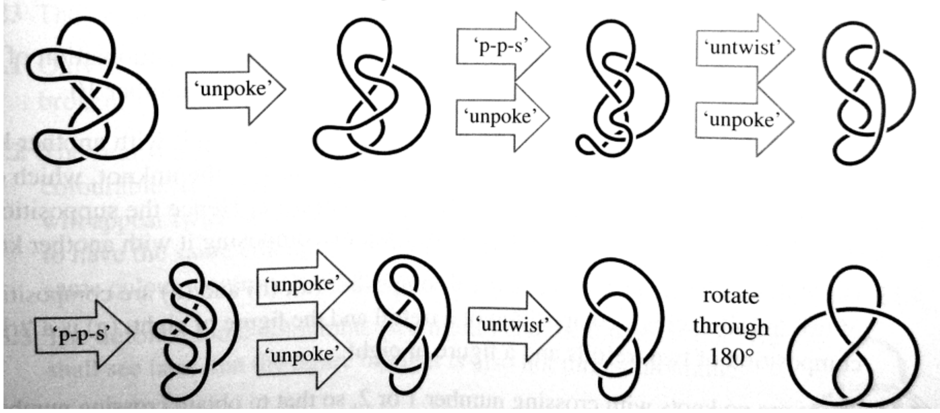


For 2b), we have the following sequence, taken from *Knots Unravelled (M. Akveld, A. Jobbings).*

By applying two 'pokes' and a 'slide' we can slide a strand from one side of a crossing to the other:

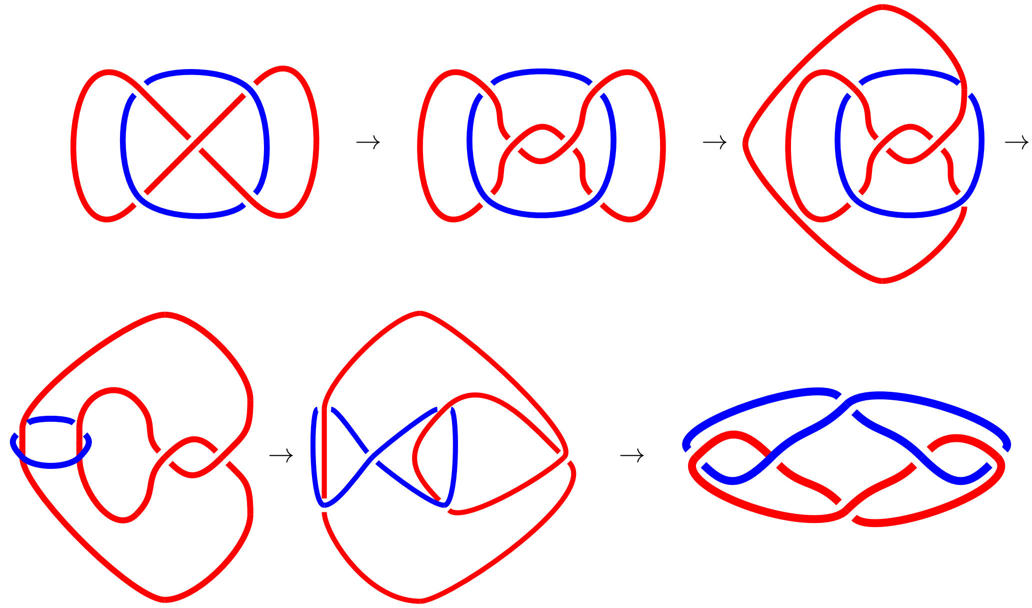


We shall want to use this procedure again, so let us use the term 'p-p-s' for the sequence of moves 'poke'-'poke'-'slide' applied in this way. In other words, 'p-p-s' moves a strand across a crossing.



For 2c), we look at the following sequence, which transforms the first version of the Whitehead link into a link for which we can exchange the two components by a 180 degree rotation around the midpoint. (Note that these transformations are already a composition of Reidemeister moves. Writing down every intermediate

step would lead to a very long sequence.)



Solution to exercise 3:

This follows from the fact that there is an unknotting sequence for any knot diagram, which we have shown in problem sheet 1. (Remember: We start at one chosen point and traverse the knot, whenever we come to a crossing the first time, we choose the overcrossing. This produces a diagram of the unknot)

Solution to exercise 4:

(a) The Jordan curve theorem implies that two distinct components in a diagram for L intersect an even number of times. Hence we add up an even number of ± 1's in the computation of $lk(L)$, which yields an even number and hence an integral linking number.

(b) Since the components of the Borromean rings are pairwise unlinked we have that the total linking number is zero for any chosen combination of orientations.

Solution to exercise 5:

Let D_1 and D_2 be diagrams for K_1 and K_2 with $c(D_1) = c(K_1)$ and $c(D_2) = c(K_2)$. Then, by construction of the connected sum, we have

$$c(K_1 \# K_2) \leq c(D_1 \# D_2) = c(D_1) + c(D_2) = c(K_1) + c(K_2)$$

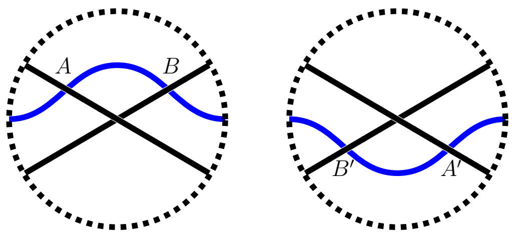
and hence we get the desired inequality for the crossing number.

For the second part let D_1 and D_2 be diagrams for K_1 and K_2 with $c(D_1) = c(K_1)$ and $c(D_2) = c(K_2)$. To obtain an unknotting sequence for the diagram $D_1\#D_2$ we first apply the operations realizing the unknotting number $u(D_1)$ to the crossings coming from D_1. Then we do the same thing for the crossings coming from D_2. This shows that

$$u(K_1\#K_2) \leq u(D_1\#D_2) \leq u(D_1) + u(D_2) = u(K_1) + u(K_2).$$

Solution to exercise 6:

For the following situation



the statement follows from these two observations:

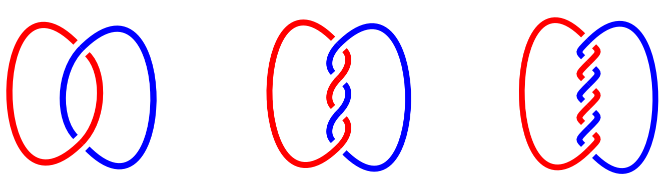
(i) The crossing A involves strands from different components of the link, if and only if the crossing A' does so. Likewise for B and B'.

(ii) For any orientation of the strands the crossing A is a positive (negative) if and only if A' is positive (negative). Likewise for B and B'

One moment's thought shows that the same argument applies to the other forms of a Reidemeister III move.

Solution to exercise 7:

The following links have linking numbers $1, 2, 3$ or $-1, -2, -3$ depending on the orientations. Continue the sequence in the obvious way to obtain a link with linking number n. (source: Wikipedia)



Solution to exercise 8:

Consider a crossing between two components of a link. Changing the orientation of one of those two components changes a right-hand-crossing to a left-hand-crossing and a left-hand-crossing to a right-hand-crossing. This happens for every crossing between two components of the link and hence every $+1$ becomes a -1 and every -1 becomes a $+1$. Hence, the linking number is negated in this case.

For the mirror image, the orientations are preserved, but every over-crossing becomes an under-crossing and every under-crossing becomes an over-crossing, i.e. again, a right-hand-crossing becomes a left-hand-crossing and a left-hand-crossing becomes a right-hand-crossing. Hence, the linking number is negated, i.e. $lk(\overline{L}) = -lk(L)$.