Solutions to sheet 4

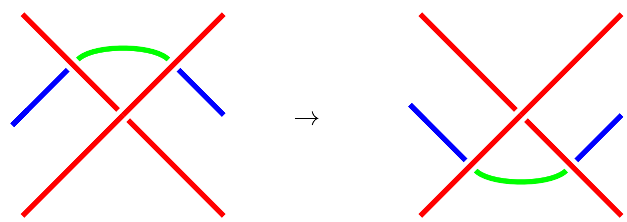
Solution to exercise 1:

There are always 3 monochromatic 3-colorings.

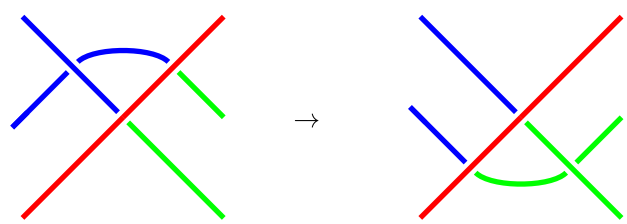
Solution to exercise 2:

We consider the colourings of the diagram locally, i.e. we only look at the strands and crossings relevant to a RIII move. Apparently the number of locally monochromatic colourings is invariant under RIII. We show that locally non-monochromatic colourings before and after the move are in 1-1 correspondence by analyzing all possible situations, i.e. we check that all diagrams satisfy the rules for 3-colourability and that the RHS is uniquely determined by the LHS and vice versa.

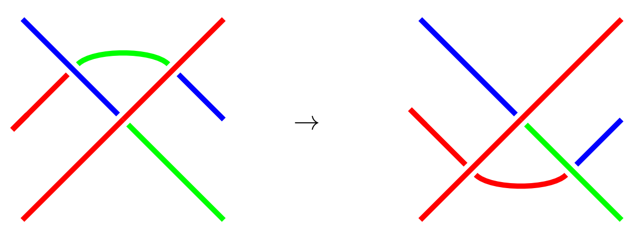
- Situation 1:



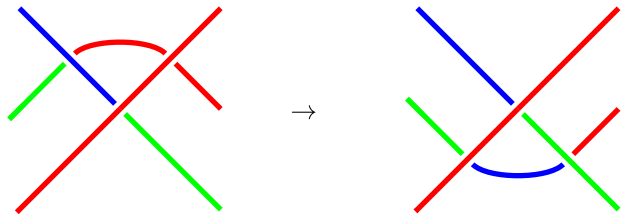
- Situation 2:



- Situation 3:



- Situation 4:



Solution to exercise 3:

The unknot has $\tau_3 = 3$. For the knot 5_1 we obtain the following matrix

$$A_1 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix} \in (\mathbb{F}_3)^{5\times 5}$$

After row reducing it in $\mathbb{F}_3$, A_1 is of the form:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in (\mathbb{F}_3)^{5\times 5}$$

For the knot 5_2 we have the matrix

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \in (\mathbb{F}_3)^{5\times 5}$$

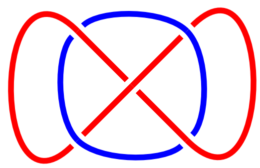
and after row reducing it in $\mathbb{F}_3$, we have

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in (\mathbb{F}_3)^{5\times 5}$$

We have $\text{rank}(A_1) = \text{rank}(A_2) = 4$ and $\text{nullity}(A_1) = \text{nullity}(A_2) = 1$, so $\tau_3(5_1) = \tau_3(5_2) = 3^1 = 3$. This means that τ_3 cannot distinguish any two of 5_1, 5_2 and the unknot.

Solution to exercise 4:

For the Whitehead link W we consider



The linking number is equal to zero (choose any orientation), as it is for the unlink. To compute $\tau_3(W)$ we observe that the corresponding matrix

$$B = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \in (\mathbb{F}_3)^{5\times 5}$$

and after row reducing it in $\mathbb{F}_3$, we obtain:

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in (\mathbb{F}_3)^{5\times 5}$$

Hence, B has $\text{nullity}(B) = 1$, so we have $\tau_3(W) = 3^1 = 3 \neq 9 = \tau_3(\text{unlink})$.

Solution to exercise 5:

(a) The linear algebra approach to the computation of $\tau_3(D)$ for a diagram D for K shows that the set of 3-colourings $V_3(D)$ is a vector space over the finite field $\mathbb{F}_3$. This implies that

$$\tau_3(K) = \tau_3(D) = |V_3(D)| = 3^{\dim V_3(D)}$$

which is a power of 3 and hence divisible by 3.

(b) see (a)

(c) Let D be a diagram for K. Think of $\tau_3(K) - 3 = \tau_3(D) - 3$ as the number of 3-colourings that are not monochromatic. Now for any such coloring we obtain another such coloring by permuting the 3 colors. Since there are 6 permutations the statement follows. In other words: The symmetric group Sym_3 acts freely on the set of non-monochromatic colourings of D, hence we get a decomposition of this set into orbits of cardinality $|\text{Sym}_3| = 6$.

Solution to exercise 6:

If D is a diagram for the knot K then we can obtain a diagram $\overline{D}$ for the mirror image $\overline{K}$ by reflecting the diagram along a line, and this reflection establishes a correspondence between 3-colourings of D and 3-colourings of $\overline{D}$.

Solution to exercise 7:

Write $K^{\#1} := K$ and $K^{\#n} := K^{\#(n-1)} \# K$. From the given identity we deduce inductively that

$$\tau_3(K^{\#n}) = \frac{1}{3^{n-1}}\, \tau_3(K)^n$$

Now pick a knot K with $\tau_3(K) > 3$, e.g. the trefoil knot. We see that

$$\tau_3(K) < \tau_3(K^{\#2}) < \tau_3(K^{\#3}) < \dots$$

Since τ_3 is an invariant we have the infinite sequence $\{K^{\#n}\}_{n\geq 1}$ of pairwise non-equivalent knots.

Solution to exercise 8:

In a colouring of the Hopf Link H both strands must have the same color, so every coloring of H is monochromatic, i.e. $\tau_3(H) = 3$.

For the other link L we need to compute τ_3 using the corresponding matrix, which is given by:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \in (\mathbb{F}_3)^{5\times 5}$$

and after row reducing it in $\mathbb{F}_3$, we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in (\mathbb{F}_3)^{5\times 5}$$

and hence $\tau_3(L) = 3$ and so we are not able to distinguish the two links using only τ_3.

Solution to exercise 9:

(a) False. By problem 3 we have $\tau_3(\text{unknot}) = \tau_3(5_1) = 0$ but the knot 5_1 is not trivial.

(b) True. τ_3 is an invariant.

(c) False. See problem 4.

(d) True. See problem 5(b).

(e) False. See problem 6 and of course $\tau_3(K) \geq 0$ by definition. (i.e. number of colourings)

(f) False. The number of crossings for two diagrams representing the same knot may be different.

(g) True. It is sufficient to show that the number of knots with crossing number *exactly* n is finite. There are, up to isotopy, only finitely many possibilities of connecting n crossings by strands (without introducing additional crossings) to form a diagram.

(h) False. Consider the sequence of knots $K_1, K_2, \ldots$, where $K_1 = 4_1$ is the figure eight knot, K_2 is the same knot with an additional twist in the middle (in fact $K_2 = 5_2$), K_3 has two additional twists etc. Every knot in this sequence obviously has unknotting number equal to 1. Intuitively they are pairwise non-equivalent, but our only computable invariants that we have at this point, the coloring numbers τ_p, won't give us a formal proof, see problem 14(b). We will see later how the Jones polynomial or the statement of Tait's conjecture can be used instead.

(i) True. The colourability-invariants do not depend on orientations.

(j) True. See problem 11.

Solution to exercise 10:

The general equation is

$$2x_{i_1} - x_{i_2} - x_{i_3} = 0 \mod p.$$

We have $2 = -1 \mod 3$ so for $p = 3$ this reads

$$-x_{i_1} - x_{i_2} - x_{i_3} = 0 \mod 3$$

which is equivalent to

$$x_{i_1} + x_{i_2} + x_{i_3} = 0 \mod 3$$

Solution to exercise 11:

In $\mathbb{Z}/2\mathbb{Z}$ the equation $2x_1 - x_2 - x_3 = 0$ is equivalent to $x_2 = x_3$ which is equivalent to saying that the two strands going "under" at the crossing have the same color. This means that we have an admissible 2-coloring of the link if all strands belonging to the same component have the same color.

Solution to exercise 12:

(a) We use that the equation $2x_{i_1} - x_{i_2} - x_{i_3} = 0$ is equivalent to $x_{i_1} + 2x_{i_2} + 2x_{i_3} = 0$ in $\mathbb{F}_5$. The relevant matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 2 & 0 & 1 & 2 \\ 2 & 2 & 0 & 1 \\ 0 & 1 & 2 & 2 \end{pmatrix}$$

has the row reduced form:

$$A = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and hence has $\text{rank}(A) = \text{nullity}(A) = 2$, so $\tau_5(4_1) = 5^2 = 25 \neq 5 = \tau_5(\text{unknot})$.

(b) We have the general identity $\tau_p(K\#K') = \frac{1}{p}\tau_p(K)\tau_p(K')$. To see this let D and D' be diagrams for K and K' with enumerated strands. We obtain a diagram for the connected sum by glueing the i-th strand of D with the j-th strand of D'. Now let V and V' be the $\mathbb{F}_p$-vector spaces of p-colourings for the two knots. We see that the space of p-colourings for $K\#K'$ is isomorphic to the space

$$S = \{(x, x') \in V \times V' \,|\, x_i = x'_j\} \subset V \times V'$$

for which we have $\dim(S) = \dim(V) + \dim(V') - 1$, so $\tau_p(K\#K') = p^{\dim(S)} = \frac{1}{p}\tau_p(K)\tau_p(K')$. In particular we have $\tau_5(4_1\#4_1) = 125$.

(c) If we enumerate the strands nicely from the left to the right we get the system of equations

$$-x_i + 2x_{i+1} - x_{i+2} = 0, \qquad 1 \leq i \leq 5$$

for the colors $x_1, \ldots, x_7 \in \mathbb{F}_5$. Solving recursively yields

$$\begin{aligned} x_5 &= 2x_6 - x_7 \\ x_4 &= 2x_5 - x_6 = 3x_6 - 2x_7 \\ x_3 &= 2x_4 - x_5 = -x_6 - 3x_7 \\ x_2 &= 2x_3 - x_4 = x_7 \\ x_1 &= 2x_2 - x_3 = x_6 \end{aligned}$$

where equations $x_2 = x_7$ and $x_1 = x_6$ correspond to $a = c$ and $b = d$.

Solution to exercise 13:

We have the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

and row reducing it (over the integers) gives the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 3 & -2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 4 & -3 & 0 & 0 & -1 \\ 0 & 0 & 0 & 5 & -4 & 0 & -1 \\ 0 & 0 & 0 & 0 & 6 & -5 & -1 \\ 0 & 0 & 0 & 0 & 0 & 7 & -7 \\ 0 & 0 & 0 & 0 & 0 & -35 & 35 \end{pmatrix}$$

and hence we obtain $\mathrm{rank}_{\mathbb{F}_5}(A) = 6$ and $\mathrm{rank}_{\mathbb{F}_7}(A) = 5$. Hence $\tau_5(7_1) = 5$ and $\tau_7(7_1) = 7^2 = 49$.

Solution to exercise 14:

(a) Replacing D_+ with D_- means replacing exactly one equation in the homogeneous system whose solutions are the p-colourings. Under this operation the rank of the system matrix can either remain constant, it can decrease by 1 or it can increase by 1. The same holds for the nullity which is the dimension of the space of colourings. Since the cardinality of an d-dimensional $\mathbb{F}_p$-vector space is p^d we see that

$$\tau_p(K_+) = k \cdot \tau_p(K_-)$$

for some $k \in \{p^{-1}, 1, p\}$.

(b) Let

$$D = D_0 \to D_1 \to D_2 \to \cdots \to D_n = \text{diagram of the unknot}$$

be a shortest unknotting sequence for K, realized on a diagram D. By repeatedly applying the result from part (a) we obtain the estimates

$$p^{-n}\tau_p(D_0) \leq \tau_p(D_n) \leq p^n \tau_p(D_0)$$

Taking the logarithm with basis p turns the left inequality into

$$\log_p(\tau_p(D_0)) - n \leq \log_p(\tau_p(D_n)) = 1$$

where we have used the fact that $\tau_p(D_n) = p$ since D_n represents the unknot. After rearranging and using $\tau_p(D_0) = \tau_p(K)$ and $u(K) = n$ we obtain the desired estimate

$$u(K) \geq \log_p(\tau_p(K)) - 1.$$