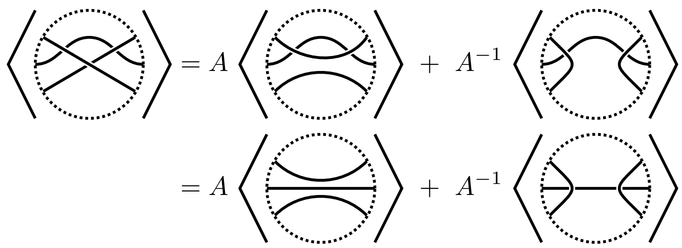
Solutions to sheet 5

Solution to exercise 1:

We have seen in the lecture that the Kauffman bracket is invariant under Reidemeister move 2. In particular, we have chosen the values in the skein relation accordingly (remember the knot algebra we did). With this, we can easily show the invariance of Reidemeister move 3 by:



Note that in the last expression above, both diagrams can be rotated by 180 degree without changing them and hence, this shows invariance under Reidemeister move 3.

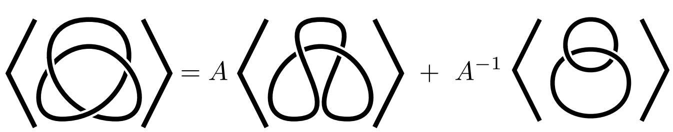
Solution to exercise 2:

We have the rules $\langle \bigcirc \rangle = 1$ and $\langle L \sqcup \bigcirc \rangle = (-A^2 - A^{-2})\langle L \rangle$, so that

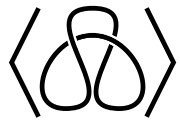
$$\begin{aligned} \langle \sqcup^n \bigcirc \rangle &= (-A^2 - A^{-2})\langle \sqcup^{n-1} \bigcirc \rangle \\ &= (-A^2 - A^{-2})^2 \langle \sqcup^{n-2} \bigcirc \rangle \\ &= \dots \\ &= (-A^2 - A^{-2})^{n-1} \langle \bigcirc \rangle \\ &= (-A^2 - A^{-2})^{n-1} \end{aligned}$$

Solution to exercise 3:

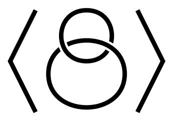
We start with the skein relation for one crossing of the trefoil, which gives:



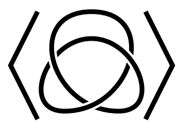
Recall that the Kauffman bracket is not invariant under Reidemeister move 1. More precisely, we have to multiply the Kauffman bracket by $-A^3$ when performing the Reidemeister move 1. Hence, the first bracket above is just $(-A^3)(-A^3)$ times the bracket of the unknot, which is 1 by definition. So, we have

A  $= A^7.$

Moreover, we have already seen in the lecture that the Kauffman bracket of the Hopf link is given by $-A^4 - A^{-4}$, hence

A^{-1}  $= -A^3 - A^{-5}.$

The Kauffman bracket of the trefoil is then

 $= A^7 - A^3 - A^{-5}.$

Solution to exercise 4:

(a) We have to show that the writhe is invariant under Reidemeister moves two (RII) and three (RIII). Let's start with RII: For any orientation of the strands we have two crossings with opposite signs before the move (i.e. that does not contribute to the writhe), and no crossing after the move. The writhe is therefore unchanged under RII. For RIII: For any orientation of the strands, the two crossings of the horizontal strand are replaced by two crossings with the same signs. The middle crossing is not affected, so the writhe is unchanged.

(b) Let D be a link diagram with c crossings and denote by w the writhe of D. Let c_1 be the number of positive crossings (i.e. those which contribute $+1$ to the writhe) and c_2 the number of negative crossings (i.e. those which contribute -1 to the writhe). We have $w = c_1 - c_2$ and $c = c_1 + c_2$. Combining those two equations give

$$c - w = (c_1 + c_2) - (c_1 - c_2) = 2c_2,$$

which means that c and w are either both even or both odd.

Solution to exercise 5:

Let T (trefoil) and H (Hopf link) be the two diagrams. Let T' be the diagram with reversed orientation and let H', H'' be the diagrams with one resp. two reversed components. We have:

$$w(T) = w(T') = -3,\ w(H) = w(H'') = 2,\ w(H') = -2$$

From problem 3 we have the result $\langle T\rangle = A^7 - A^3 - A^{-5}$ as well as the intermediate result $\langle H\rangle = -A^4 - A^{-4}$ (from the lecture). We obtain the following X-polynomials

$$\begin{aligned} X(T) = X(T') &= (-A)^9(A^7 - A^3 - A^{-5}) = -A^{16} + A^{12} + A^4 \\ X(H) = X(H'') &= (-A)^{-6}(-A^4 - A^{-4}) = -A^{-2} - A^{-10} \\ X(H') &= (-A)^6(-A^4 - A^{-4}) = -A^{10} - A^2. \end{aligned}$$

Solution to exercise 6:

Using the result of the previous problem we have

$$\begin{aligned} V(3_1) &= X(3_1)\big|_{A=t^{-\frac{1}{4}}} \\ &= -A^{16} + A^{12} + A^4\big|_{A=t^{-\frac{1}{4}}} \\ &= -t^{-4} + t^{-3} + t^{-1}. \end{aligned}$$

Changing the orientation has no influence on $V(3_1)$ by the previous problem.

Solution to exercise 7:

Let c be the number of crossings in a given diagram.

(a) The writhe is minimal if all crossings are negative and maximal if all crossings are positive. Hence, $-c \leq \sum s \leq c$.

(b) Same proof as in Exercise 4b).

Solution to exercise 8:

If D is an oriented knot diagram with reverse D' then $w(D) = w(D')$, since at every crossing, both directions change, i.e. every right-handed crossings stays right-handed and every left-handed crossing stays left-handed. Since the Kauffman bracket does not depend on orientations we have $V(D) = V(D')$.