Solutions to sheet 6

Solution to exercise 1:

The state sum formula is

$$\langle D\rangle = \sum_s A^{\sum s} P^{|sD|-1}$$

where $P := -A^2 - A^{-2}$. Let's make the convention that a diagram without crossings has a single state s such that $sD = D$ and $\sum s$=0. (Recall that $|sD|$ is the number of disjoint loops.)

The above formula holds true when D is a diagram without crossings, i.e. the trivial link with n components (as seen in exercise sheet 5, exercise 2). Now we can prove the formula for general diagrams by induction on the number of crossings.

Let D be a diagram which we resolve at the i-th crossing so that $\langle D\rangle = A\langle D_+\rangle + A^{-1}\langle D_-\rangle$. By induction hypothesis we have the following equalities:

$$\begin{aligned} A\langle D_+\rangle &= A \cdot \sum_{s \text{ state of } D_+} A^{\sum s} P^{|sD_+|-1} \\ &= \sum_{s \text{ state of } D_+} A^{\sum s+1} P^{|sD_+|-1} \\ &= \sum_{\substack{s \text{ state of } D \\ \text{with } s_i \text{ a positive crossing}}} A^{\sum s} P^{|sD|-1} \end{aligned}$$

Similar, we have

$$\begin{aligned} A^{-1}\langle D_+\rangle &= A^{-1} \cdot \sum_{s \text{ state of } D_-} A^{\sum s} P^{|sD_-|-1} \\ &= \sum_{s \text{ state of } D_-} A^{\sum s-1} P^{|sD_-|-1} \\ &= \sum_{\substack{s \text{ state of } D \\ \text{with } s_i \text{ a negative crossing}}} A^{\sum s} P^{|sD|-1} \end{aligned}$$

And hence

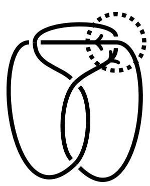
$$\begin{aligned} \langle D\rangle &= A\langle D_+\rangle + A^{-1}\langle D_-\rangle \\ &= \sum_{\substack{s \text{ state of } D \\ \text{with } s_i \text{ a positive crossing}}} A^{\sum s} P^{|sD|-1} + \sum_{\substack{s \text{ state of } D \\ \text{with } s_i \text{ a negative crossing}}} A^{\sum s} P^{|sD|-1} \\ &= \sum_{s \text{ state of } D} A^{\sum s} P^{|sD|-1} \end{aligned}$$

which means that the state sum formula holds true for the diagram D.

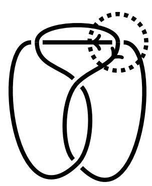
Solution to exercise 2:

- The Kauffman bracket of the figure-eight-knot is given by $A^{-8} - A^{-4} + 1 - A^4 + A^8$ and the writhe is zero. Substituting $A \to t^{-\frac{1}{4}}$ gives the Jones polynomial
$$V_{4_1}(t) = t^{-2} - t^{-1} + 1 - t + t^2$$
- We use the skein relation
$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{L_0}(t)$$
where L_+, L_- and L_0 are chosen as

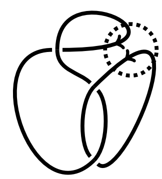
$L_+ = 4_1$:



$L-$:



L_0 :

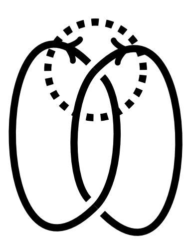


We know that $V_{L_-} = 1$, as it is an unknot and $V_{L_0} = -t^{5/2} - t^{1/2}$ is the Hopf link (see exercise 3a)). Hence
$$\begin{aligned} V(L_+) &= t^2 V(L_-) + (t^{3/2} - t^{1/2})V(L_0) \\ &= t^{-2} - t^{-1} + 1 - t + t^2. \end{aligned}$$

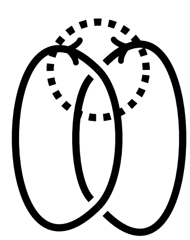
Solution to exercise 3:

(a) We use the skein relation
$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V_{L_0}(t)$$
to calculate the Jones polynomial of the Hopf link L. We use

$L_+ = L$:



$L-$:



and L_0 is the unknot. L_- is the unlink with two components, i.e. we have $V_{L-}(t) = (-t^{1/2} - t^{-1/2})$ (for any orientation of the components). So we obtain

$$V_{L_+} = -t^{5/2} - t^{1/2}$$

Since this polynomial is not invariant under $t \mapsto t^{-1}$ we deduce that the Hopf link is not equivalent to its mirror image.

(b) Similarly, we can view the trefoil as L_+ and resolve one crossing to obtain L_-, which is the unknot and L_0, which is the negative Hopf link. We get:

$$\begin{aligned} V(L_+) &= t^2 V(L_-) + (t^{3/2} - t^{1/2}) V(L_0) \\ &= t + t^3 - t^4. \end{aligned}$$

Solution to exercise 4:

(a) Let D_1, D_2 be oriented diagrams for K_1, K_2. Observe that every state s for $D := D_1 \# D_2$ corresponds to a pair (s_1, s_2), where s_i is a state for D_i. We write $s = (s_1, s_2)$. We have $\sum s = \sum s_1 + \sum s_2$ and $|sD| = |s_1 D_1| + |s_2 D_2| - 1$. We use these facts for the following computation, where $P = -A^2 - A^{-2}$:

$$\begin{aligned} \langle D_1 \# D_2 \rangle &= \sum_s A^{\sum s} P^{|sD|-1} \\ &= \sum_{(s_1,s_2)} A^{\sum s_1 + \sum s_2} P^{|s_1 D| + |s_2 D| - 2} \\ &= \sum_{(s_1,s_2)} A^{\sum s_1} P^{|s_1 D|-1} A^{\sum s_2} P^{|s_2 D|-1} \\ &= \left(\sum_{s_1} A^{\sum s_1} P^{|s_1 D|-1} \right) \left(\sum_{s_2} A^{\sum s_2} P^{|s_2 D|-1} \right) \\ &= \langle D_1 \rangle \langle D_2 \rangle. \end{aligned}$$

Now the fact that the writhe satisfies $w(D_1 \# D_2) = w(D_1) + w(D_2)$ yields

$$\begin{aligned} X(K_1 \# K_2) &= (-A)^{-3w(K_1 \# K_2)} \langle K_1 \# K_2 \rangle \\ &= (-A)^{-3w(K_1)} \langle K_1 \rangle (-A)^{-3w(K_2)} \langle K_2 \rangle \\ &= X(K_1) X(K_2) \end{aligned}$$

and after the substitution $A \mapsto t^{-1/4}$ we have the desired identity.

(b) The argument of part (a) needs only a slight modification in order to work here as well. Namely, for $s = (s_1, s_2)$ we now have $|sD| = |sD_1| + |sD_2|$. The result of the above computation then has an additional factor $P = -A^2 - A^{-2} = -t^{1/2} - t^{-1/2}$. Alternatively, one can apply the skein relation to a knot L for which $L_+ = L_- = K_1 \# K_2$ and $L_0 = K_1 \sqcup K_2$.

Solution to exercise 5:

Choose an orientation for the diagram. We apply the skein relation to the crossing below the top crossing. This is a left-hand crossing, hence the given diagram is L_+. One checks that L_- is the left-hand trefoil knot and L_0 is a positive Hopf link. Using results from previous problems we obtain

$$\begin{aligned} V(L_-) &= t^{-2}V(L_+) + (t^{-3/2} - t^{-1/2})V(L_0) \\ &= t^{-2}(-t^{-4} + t^{-3} + t^{-1}) + (t^{-3/2} - t^{-1/2})(-t^{-5/2} - t^{-1/2}) \\ &= -t^{-6} + t^{-5} - t^{-4} + 2t^{-3} - t^{-2} + t^{-1}. \end{aligned}$$