Solutions to sheet 7

Solution to exercise 1:

(a) Suppose L_+ has μ components. Then L_- has μ components as well, since no crossing is removed or added, only one over-crossing turns into an under-crossing or vice versa. For L_0, we have two possibilities, depending on how we connect the strings when removing a crossing. Either we connect two components of L_+ or we separate one component of L_+. Hence, L_0 has either $\mu + 1$ or $\mu - 1$ components.

(b) We proceed by induction on the complexity of a knot diagram D. Recall that the complexity was given by a pair (c, m), where c denotes the number of crossings in D and m the number of crossings needed to change D into a diagram of the unknot. The order is lexicographical. So, let us start with a diagram of complexity $(0, 0)$, i.e. the trivial link with μ components. For $\mu = 1$, we have a diagram of the unknot and hence $V(unknot) = 1$, which has only integral powers in t and t^{-1}. We have seen on the last exercise sheet that

$$V(\cup^{\mu}\ unknot) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^{\mu-1}$$

which is a polynomial with integral powers in t and t^{-1} if μ is odd and a polynomial with half-integral powers in t and t^{-1} if μ is even. That shows the induction start.

For the induction step, recall the following Lemma from the lecture: *"Given an oriented link L with complexity (c, m), there exists a diagram D and a choice of crossing C so that the three links L_+, L_- and L_0 associated to D and C are L together with two new links of complexity strictly lower than (c, m)".*

Let us consider a diagram D with complexity (c, m). Let L_+ be the diagram corresponding to D and L_- and L_0 both of complexity lower than (c, m), by the lemma above (The other cases, i.e. for L_- corresponding to D or L_0 corresponding to D, can be treated similarly). Recall the Jones skein relation

$$t^{-1}V(L_+) - tV(L_-) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V(L_0),$$

which can be rewritten as

$$V(L_+) = t^2V(L_-) + (t^{\frac{3}{2}} - t^{\frac{1}{2}})V(L_0). \qquad (1)$$

Now, assume that the number of components μ of L_+ is even. By a), we know that the number of components of L_- is μ as well, hence even, and the number of components of L_0 is $\mu + 1$ or $\mu - 1$, hence odd. By the induction hypothesis, $V(L_-)$ consists of half integral powers of t and t^{-1} only and $V(L_0)$ consists of integral powers of t and t^{-1} only. Hence by (1), $V(L_+)$ consists of half integral powers of t and t^{-1} only.

Analogously for μ odd, which finishes the proof.

Solution to exercise 2:

Take an alternating diagram and insert two consecutive under-crossings by performing a Reidemeister move II anywhere in the diagram.

Solution to exercise 3:

That problem is still open.

Solution to exercise 4:

Recall that the Kauffman bracket is invariant under Reidemeister move 2 and 3. Only for Reidemeister move 1, we had to multiply the whole bracket by $-A^3$. Since multiplying the whole polynomial by $-A^3$ does not change the span (which is the difference between the highest and the lowest power), the span of the Kauffman bracket is actually invariant under Reidemeister 1 as well and hence a knot invariant.

Solution to exercise 5:

Let D be a reduced alternating diagram for K. From the lecture we know that the diagram is minimal, i.e. $c(D) = c(K)$. Moreover, we know that $\mathcal{B}(\langle D \rangle) = 4c(D)$. So we have

$$\mathcal{B}(V(K)) = \frac{1}{4}\mathcal{B}(X(K)) = \frac{1}{4}\mathcal{B}(\langle D \rangle) = \frac{1}{4} \cdot 4c(D) = c(D) = c(K).$$

Solution to exercise 6:

We claim that the knot $K_1 \# K_2$ is alternating. To see this, take alternating diagrams D_1 and D_2 for K_1 and K_2. Put them next to each other. There are two ways to obtain the connected some, as one diagram can be 'flipped' (i.e. turned 180 degrees). Both diagrams represent the connected sum $K_1 \# K_2$ and exactly one of them is alternating. Now we have, by the previous exercise:

$$\begin{aligned} c(K_1 \# K_2) &= \mathcal{B}(V(K_1 \# K_2)) \\ &= \mathcal{B}(V(K_1)V(K_2)) \\ &= \mathcal{B}(V(K_1)) + \mathcal{B}(V(K_2)) \\ &= c(K_1) + c(K_2). \end{aligned}$$

Solution to exercise 7:

Take any n-crossing diagram of the unknot, i.e. perform for example n Reidemeister moves 1. by Exercise 4, the span of the Kauffman bracket is a knot invariant and hence, it is zero for this diagram, since the span of the Kauffman bracket of the unknot is zero as well.

Solution to exercise 8:

Since K has a reduced alternating diagram with n crossings, it follows by the Tait conjecture that $n = c(K)$.

(a) Assume that K is equivalent to its mirror image. This implies that the Jones polynomial $V(K)$ is invariant under the involution $t \mapsto t^{-1}$, hence it has the form

$$V(K) = \alpha_0 + \alpha_1(t + t^{-1}) + \cdots + \alpha_k(t^k + t^{-k})$$

with $\alpha_k \neq 0$. In particular, we have that $n = c(K) = \mathcal{B}(V(K)) = k - (-k) = 2k$ is even, which contradicts our assumption.

(b) We have $V(K\#K) = V(K)^2$, so the Jones polynomials have the form

$$V(K) = \alpha_l t^l + \cdots + \alpha_k t^k$$

and

$$V(K\#K) = \alpha_l^2 t^{2l} + \cdots + \alpha_k^2 t^{2k}$$

Now assume that $K\#K$ is equivalent to its mirror image. Since $V(K\#K)$ is invariant under $t \mapsto t^{-1}$ we have $-2l = 2k$, i.e. $-l = k$ and hence the span of $V(K)$ is equal to $2k$, so as above $2k = \mathcal{B}(V(K)) = c(K) = n$, which is a contradiction. This means that $K\#K$ cannot be equivalent to its mirror image if K has odd crossing number.

Solution to exercise 9:

By exercise 5 we have $\mathcal{B}(V(8_i)) = 8$ if the knot 8_i is alternating. For $i \in \{19, 20, 21\}$ the span of the Jones polynomial is different from 8 (which one can check in a knot table), and hence these knots are not alternating.