Solutions to sheet 8

Solution to exercise 1:

Let H^+ be the positive Hopf link, i.e. the one with linking number +1. Consider L_+, L_-, L_0 such that $L_+ = H^+$. Then L_- is the trivial link with two components and L_0 is the unknot. To compute $P(L_-)$ consider S_0, S_+, S_- with $S_0 = L_-$. We have $S_+ = S_- = O$, hence the skein relation says

$$l \cdot P(L_+) + l^{-1} \cdot P(L_-) = -m \cdot P(L_0)$$

which implies $P(L_-) = -m^{-1}(l + l^{-1})$. Applying the skein relation again for the Hopf link, we get

$$P(H^+) = m^{-1}(l^{-1} + l^{-3}) - ml^{-1}.$$

We will also need the HOMFLY polynomial for the negative Hopf link H^-, which is

$$P(H^-) = m^{-1}(l + l^3) - ml.$$

Solution to exercise 2:

Let K be the left-hand trefoil and let K_+, K_-, K_0 such that $K_- = K$. Then we have K_+ the unknot and K_0 the negative Hopf link. Hence the skein relation reads

$$l \cdot P(unknot) + l^{-1} \cdot P(3_1) + m \cdot P(H^-) = 0$$

which, using the result from the previous problem, yields

$$P(K) = -2l^2 - l^4 + m^2 l^2.$$

Solution to exercise 3:

This follows from the fact that a crossing in an oriented knot K is positive/negative if and only if the same crossing is positive/negative for the reverse knot rK. This means that the skein relation looks the same for K and rK, so $P(K) = P(rK)$.

Solution to exercise 4:

(a) The skein relation says that

$$l \cdot P(L^+) + l^{-1} \cdot P(L^-) + m \cdot P(L_0) = 0.$$

where we take $L_0 = L \cup O$ and it turns out that $L_+ = L_- = L$, as Reidemeister move 1 (and its reversion) removes the part coming from the unknot in L_+ and L_-. Hence, we have

$$P(L \sqcup O) = m^{-1}(l + l^{-1})P(L).$$

(b) We deduce this statement from (c). We have $L^+ = L^- = L_1 \# L_2$ and $L_0 = L_1 \cup L_2$, so applying (c) gives $P(L^+) = P(L^-) = P(L_1)P(L_2)$. The skein relation implies

$$P(L_1 \sqcup L_2) = -(l + l^{-1})m^{-1}P(L_1)P(L_2).$$

(c) We prove the statement by induction on the crossing number $c(L_2)$. If $c(L_2) = 0$ then L_2 is the unknot and $L_1 \# L_2 = L_1$, so the statement holds true since $P(O) = 1$. Now assume that $c(L_2) = n > 0$. Take a minimal diagram for L_2. Since it has an unknotting sequence we can pick a crossing such that the crossing number gets smaller if we change over- and undercrossing. Assume that there is a positive crossing with this property. We write $L_2^+ = L_2$ and consider the usual local variations L_2^- and L_2^0 at the crossing under consideration. We have $c(L_2^-), c(L_2^0) \leq n-1$, so by induction hypothesis

$$P(L_1 \# L_2^-) = P(L_1)P(L_2^-), \qquad P(L_1 \# L_2^0) = P(L_1)P(L_2^-)$$

and hence the skein relation

$$l \cdot P(L_1 \# L_2^+) + l^{-1} \cdot P(L_1 \# L_2^-) + m \cdot P(L_1 \# L_2^0) = 0$$

implies

$$\begin{aligned} l \cdot P(L_1 \# L_2) &= l \cdot P(L_1 \# L_2^+) \\ &= -l^{-1} \cdot P(L_1 \# L_2^-) - m \cdot P(L_1 \# L_2^0) \\ &= P(L_1)(-l^{-1} \cdot P(L_2^-) - m \cdot P(L_2^0)) \\ &= l \cdot P(L_1)P(L_2^+) \\ &= l \cdot P(L_1)P(L_2) \end{aligned}$$

which implies the statement. If we can't find an unknotting sequence for L_2 which begins with a positive crossing then we pick a negative crossing and set $L_2^- = L_2$ so that we can apply the same argument.

Solution to exercise 5:

We have two possible ways of connecting L_1 with L_2, namely (i) we can connect L_2 with the trefoil component of L_1 or (ii) with the trivial component of L_1. These two possibilities yield non-equivalent links $R_{(i)} = L_1 \#_{(i)} L_2$, $R_{(ii)} = L_1 \#_{(ii)} L_2$. However, by part (c) of the previous problem we have $P(R_{(i)}) = P(R_{(ii)}) = P(L_1)P(L_2)$.

Solution to exercise 6:

The substitution is

$$l = it^{-1}, \qquad m = i(t^{-1/2} - t^{1/2}).$$

For an oriented link L let $P'(L) \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$ be the polynomial obtained by performing this substitution in the HOMFLY polynomial $P(L)$. The skein relation for P says

$$it^{-1}P'(L_+) - itP'(L_-) + i(t^{-1/2} - t^{1/2})P'(L_0) = 0,$$

i.e.

$$t^{-1}P'(L_+) - tP'(L_-) + (t^{-1/2} - t^{1/2})P'(L_0) = 0.$$

Furthermore we have $P'(O) = P(O) = 1$. We have seen that this skein relation together with the normalization on the unknot defines the Jones polynomial V uniquely, i.e. we have $P' = V$.

Solution to exercise 7:

In class we proved that for a two-component link L we have $a_1(L) = \mathrm{lk}(L)$. We assume here that L_+, L_- have one component, so L_0 has two components and therefore $a_1(L_0) = \mathrm{lk}(L_0)$. The skein relation for the Conway polynomial says that

$$\nabla_{L_0}(z) = \frac{1}{z}\left(\nabla_{L_+}(z) - \nabla_{L_-}(z)\right)$$

so by comparison of coefficients we get

$$\mathrm{lk}(L_0) = a_1(L_0) = a_2(L_+) - a_2(L_-).$$

Solution to exercise 8:

From Exercise 2 we know that the HOMFLY-polynomial of 3_1 is given by:

$$P(3_1) = -2l^2 - l^4 + m^2l^2$$

The Alexander polynomial is then obtained by taking $l = i$ and $m = i(t^{-\frac{1}{2}} - t^{\frac{1}{2}})$, where $i = \sqrt{-1}$, so we get:

$$\Delta_{3_1}(t) = 1 + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2$$

The Conway polynomial is obtained by taking $z \mapsto (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$, i.e. we get

$$\nabla_{3_1}(z) = 1 + z^2$$

Similarly, the HOMFLY-polynomial of $\overline{3_1}$ is given by

$$P(3_1) = -2l^{-2} - l^{-4} + m^2l^{-2}$$

which leads to

$$\nabla_{\overline{3_1}}(z) = 1 + z^2$$

as well. That means that the Conway polynomial cannot distinguish the trefoil knot from its mirror image.