Solutions to sheet 9

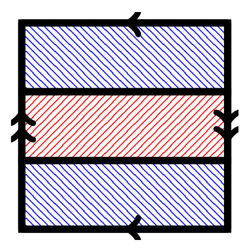
Solution to exercise 1:

(a) Let M be the Möbius strip obtained by a suitable identification of two opposite sides of the unit square $[0,1]^2$. We can identify the boundary ∂M with S^1. Consider the map $\varphi_0\colon S^1 \times [0, \frac{1}{2}] \to M$ which sends (p,t) to the point which is obtained by starting from the boundary point $p \in S^1 = \partial M$ and going distance t orthogonally into the band. This map is continuous and surjective. It is not injective, since $\varphi_0(p, \frac{1}{2}) = \varphi_0(-p, \frac{1}{2})$ for all p. This means however, that the map φ_0 descends to the quotient

$$C := (S^1 \times [0, \frac{1}{2}]) \,/\, ((p, \frac{1}{2}) \sim (-p, \frac{1}{2}))$$

which is a crosscap. So we have a map $\varphi\colon C \to M$. This map is continuous and bijective, hence a homeomorphism (since C is compact and M is Hausdorff).

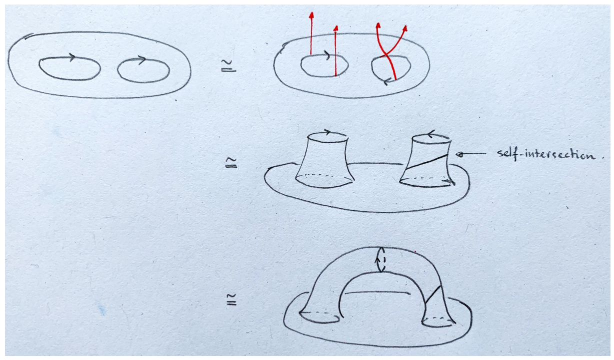
(b) The following square yields the Klein bottle when the indicated glueings are performed. By cutting and identifying the two horizontal lines as indicated, the two (!) depicted regions are then Möbius strips with common boundary.



Solution to exercise 2:

(a) Let X be a sphere with a twisted handle attached and let Y be a sphere with two crosscaps attached. We show that there is a homeomorphism $\varphi\colon X \to Y$. For an open disk $D \subset X$ we then have an induced homeomorphism $X \backslash D \to Y \backslash \varphi(D)$, where $X \backslash D$ and $Y \backslash \varphi(D)$ are exactly the spaces given in the problem. (We are using that a sphere minus a disk is another disk, and that the fact that it is irrelevant where in a surface one removes an open disk.) The space X is nothing but a Klein bottle, since a twisted handle is a Klein bottle minus a disk. To obtain the space Y we first remove two disks, which yields a cylinder. Glueing a crosscap on each of the boundary components of the cylinder is (by problem 1) the same as glueing two Möbius strips on the cylinder. This in turn is the same as glueing two Möbius strips along their boundary, which (again by problem 1) yields a Klein bottle. Hence X and Y are both Klein bottles and therefore there is a homeomorphism $\varphi\colon X \to Y$.

(b) As a warm up we notice that a twisted handle can be realized in three dimensional space by first ”self-intersecting” one of the two parts in order to have the two boundary components oriented in the same direction:



Once this 3-dimensional construction is assimilated the proof can be found in the following paper of Francis and Weeks : `https://www.jstor.org/stable/2589143` (Lemma 3).

Solution to exercise 3:

(a) Take an annulus surrounding each disc, and define a map which is the identity on the rest of the disc, but compresses the annulus radially by a factor of 2 towards the outside. This map shrinks the 'disc with holes' into itself. Simultaneously, chop the cylinder into three segments, and map the outer ones to the 'missing' parts of the annuli (the inner halves) and stretch the middle segment to cover the whole cylinder now.

(b) In this case X is a disc with a twisted handle attached.

Solution to exercise 4:

(This is Lemma 5.4.6. in the Knotes of J. Roberts) The map is defined by re-identifying the edges in F' which we just 'unidentified'. It is a quotient map and therefore continuous. One can check that restricted to the boundary of F', the map p is a $2:1$ covering map onto C. But there are only two double covers of the circle, the connected one and the disconnected one (see for instance pp.67-68 in Hallen Hatcher's algebraic topology book: `https://pi.math.cornell.edu/~hatcher/AT/AT.pdf`).

Solution to exercise 5:

(a) Recall that a curve is called *separating* if the resulting surface F' has more components than F.

Suppose that F is connected (otherwise we restrict ourselves to the component containing C). For every point in F there is a curve joining it to C and since $p\colon F' \to F$ is the identity outside $p^{-1}(C)$, for every point in F' there is a curve joining it to $p^{-1}(C)$. It follows from the previous exercise that F' can't have more than 2 components.

(b) This follows from the fact that a one-sided curve has a neighborhood which is homeomorphic to a Möbius strip, together with the observation that a Möbius strip does not disconnect when cut a long its middle circle.