Applied Stochastic Processes

Exercise sheet 1

Exercise 1.1

(a) Let $T_1, \dots, T_k$ be i.i.d. random variables with $T_1 \sim \text{Exp}(\lambda)$ and $S_k = \sum_{i=1}^k T_i$. Show that $S_k \sim \text{Gamma}(k, \lambda)$.

(b) A real and positive random variable X is said to have the *memoryless* property if $\text{P}[X \geq x] > 0$ for all $x > 0$ and

$$\text{P}(X \geq x + y \,|X \geq x) = \text{P}(X \geq y) \text{ for all } x, y > 0.$$

Prove that a continuous and positive random variable X has the memoryless property if and only if $X \sim \text{Exp}(\lambda)$ for some $\lambda > 0$.

Exercise 1.2 Wald's Equation.
Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of i.i.d. random variables with $\text{E}[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2 < \infty$ and $(S_n)_{n \in \mathbb{N}}$ the sequence of partial sums defined by $S_0 := 0$ and $S_n := \sum_{i=1}^n X_i$.

For a non-negative, integer-valued random variable N, which is independent of $(X_i)_{i \in \mathbb{N}}$, let S_N denote the random sum defined by $S_N := \sum_{i=1}^N X_i$.

(a) Suppose that $\text{E}[N] < \infty$. Prove

$$\text{E}[S_N | N] = \mu N$$

and

$$\text{E}[S_N] = \mu \text{E}[N].$$

Hint: Do not forget to argue that S_N is integrable.

(b) Suppose that $\text{E}[N^2] < \infty$. Show that

$$\text{E}[S_N^2 | N] = \sigma^2 N + \mu^2 N^2$$

and

$$\text{Var}(S_N) = \sigma^2 \text{E}[N] + \mu^2 \text{Var}(N).$$

Exercise 1.3 Spatial Poisson process.
Let countably many points be distributed in $\mathbb{R}^2$ according to the following rule:

1. For a bounded set $A \in \mathcal{B}(\mathbb{R}^2)$ the number of points $N(A)$ lying in the set A is Poisson-distributed with parameter $\mu(A)$ where $\mu(A) := \lambda |A|$ with $\lambda > 0$ and $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^2$.
2. $N(A_1), \dots, N(A_k)$ are independent for disjoint bounded sets $A_1, \dots, A_k \in \mathcal{B}(\mathbb{R}^2)$.

(a) For fixed $r > 0$ we define $B_r := \{x \in \mathbb{R}^2 : \|x\| \leq r\}$ (where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^2$) and $D := \inf\{r > 0 \,|\, N(B_r) > 0\}$. Determine the distribution function and the density of D.

(b) Compute for $u > r$ the limit $\lim_{r \to 0} \text{P}[N(B_u) = 1 \,|\, N(B_r) = 1]$.

Submission deadline: 13:15, Feb. 28.

Location: During exercise class or in the tray outside of HG E 65.

Class assignment:

Students	Time & Date	Room	Assistant
A-K	Thu 09-10	HG D 7.2	Maximilian Nitzschner
L-Z	Thu 12-13	HG D 7.2	Daniel Contreras

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2019/fs/401-3602-00L/