Applied Stochastic Processes

Exercise sheet 10

Exercise 10.0 Frog Markov chain
Let \((X_n)_{n \geq 0}\) be the Markov chain with state space \(\{1, 2\}\), initial distribution \(\mu = (\mu_1, \mu_2)\) and transition matrix
\[
P = \begin{pmatrix} 1 - p & p \\ q & 1 - q \end{pmatrix}, \quad \text{where } 0 < p, q \leq 1\]
(a) Compute \(P\mu[X_n = i]\) for every \(n\).
(b) Deduce the value of \(\lim_{n \to \infty} P\mu[X_n = i]\).

Exercise 10.1 Let \(E\) be a countable state space and fix \(x \in E\). We consider a Markov chain \((X_n)_{n \geq 0}\) under \(P_x\). Assume that \(P_x[H^+_x < \infty] = 1\). Define \(H_x^{(1)} = H^+_x\) and
\[
H_x^{(n+1)} = \min \{k > 0; X_{T_n+k} = x\}\]
for \(n \geq 1\), where \(T_i = H_x^{(1)} + \cdots + H_x^{(i)}\).
(a) Show that under \(P_x\), the random variables \((H_x^{(i)})_{i \geq 1}\) are i.i.d.
(b) Show that the process defined by \(N_t = \sum_{1 \leq i \leq t} 1\{X_i = x\}\) is a renewal process.

Exercise 10.2 Let us consider the reflected random walk, that is, the Markov chain with state space \(\mathbb{N}_0\) and transition probability given by \(p_{0,1} = 1\) and \(p_{x,x+1} = \alpha, p_{x,x-1} = 1 - \alpha\) for \(x \geq 1\).
Show that for \(\alpha \leq 1/2\) all the states are recurrent, and for \(\alpha > 1/2\) all the states are transient.

Exercise 10.3 Snakes and ladders.
A simple game of ‘snakes and ladders’ is played on a board of nine squares.

At each turn a player tosses a fair coin and advances one or two places according to whether the coin lands heads or tails. If you land at the foot of a ladder you climb to the top, but if you land at the head of a snake you slide down to the tail.
(a) How many turns on average does it take to complete the game?
Hint: Call \(k_i = E_i[H_9]\) and find some relations between the \(k_i\) for \(i \in \{1, \ldots, 9\}\).
(b) What is the probability that a player who has reached the middle square will complete the game without slipping back to square 1?
Submission deadline: 13:15, May 9.

Location: During exercise class or in the tray outside of HG E 65.

Class assignment:

<table>
<thead>
<tr>
<th>Students</th>
<th>Time &amp; Date</th>
<th>Room</th>
<th>Assistant</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-K</td>
<td>Thu 09-10</td>
<td>HG D 7.2</td>
<td>Maximilian Nitzschner</td>
</tr>
<tr>
<td>L-Z</td>
<td>Thu 12-13</td>
<td>HG D 7.2</td>
<td>Daniel Contreras</td>
</tr>
</tbody>
</table>

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Exercise sheets and further information are also available on: http://metaphor.ethz.ch/x/2019/fs/401-3602-00L/