ETH Zürich, FS 2019
D-MATH
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Coordinator
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Applied Stochastic Processes

Exercise sheet 11

Exercise 11.1 Recurrence and stationarity.
Consider the Markov chain $(X_n)_{n\in\mathbb{N}}$ with state space $\mathbb{N} := \{1, 2, ...\}$, and transition probability

$$p_{i,j} = \begin{cases} \pi(j) & \text{if } i = 1,\ j \geq 1, \\ 1 & \text{if } i > 1,\ j = i - 1, \\ 0 & \text{else,} \end{cases}$$

where π denotes a probability distribution on $\mathbb{N}$ with $\sum_{i\in\mathbb{N}} i\pi(i) < \infty$.

(a) Determine the positive recurrent, null recurrent, and transient states.

(b) Find a stationary distribution ν for this Markov chain. Is this the unique stationary distribution?

Exercise 11.2 Let $(X_n)_{n\geq 0}$ a Markov chain on a countable state space E. Let $x, y \in E$ such that $x \longleftrightarrow y$. Prove that x is positive recurrent if and only if y is positive recurrent.

Exercise 11.3 Simple Random Walk on $\mathbb{Z}^d$ and Fourier Analysis.
Let $(X_n)_{n\geq 0}$ be the simple random walk on $\mathbb{Z}^d$, $d \geq 1$. Suppose that $X_0 = 0$ and let V_0 be the total number of returns to 0. For $\xi \in [-\pi, \pi]^d$, we denote the characteristic function X_1 by $\varphi(\xi) = \mathbf{E}_0[\exp(i\xi \cdot X_1)]$.

(a) Show that

$$\mathbf{P}_0[X_n = 0] = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \varphi(\xi)^n d\xi.$$

(b) Show that

$$\mathbf{E}_0[V_0] = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{\varphi(\xi)}{1 - \varphi(\xi)} d\xi.$$

(c) Calculate explicitly $\varphi(\xi)$ and show that

$$\frac{1}{4d}|\xi|^2 \leq 1 - \varphi(\xi) \leq \frac{1}{2d}|\xi|^2.$$

(d) Conclude that 0 is a recurrent state if and only if $d \in \{1, 2\}$.

Submission deadline: 13:15, May 16.

Location: During exercise class or in the tray outside of HG E 65.

Class assignment:

Students	Time & Date	Room	Assistant
A-K	Thu 09-10	HG D 7.2	Maximilian Nitzschner
L-Z	Thu 12-13	HG D 7.2	Daniel Contreras

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2019/fs/401-3602-00L/