Coordinator
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Applied Stochastic Processes

Exercise sheet 12

Exercise 12.1

(a) Find all the stationary probability measures of the reflected random walk on $\mathbb{Z}$.

(b) Consider the biased random walk on the circle $\mathbb{Z}/n\mathbb{Z}$, i.e. the Markov chain with transition probability given by $p_{x,x+1} = \alpha$ and $p_{x,x-1} = 1 - \alpha$ for all $x \in \mathbb{Z}/n\mathbb{Z}$. Show that there exists a reversible probability measure for this Markov chain if and only if $\alpha = 1/2$.

Exercise 12.2 Let $(X_n)_{n\geq 0}$ a Markov chain on a countable state space E. Let us denote $R_1^+, R_2^+, \ldots$ its positive recurrent classes.

(a) For $i = 1, 2, \ldots$ we consider the process $(X_n^{(i)})_{n\geq 0}$ with state space R_i^+ and transition probability given by $(p_{x,y})_{x,y\in R_i^+}$, where p is the transition probability for $(X_n)_{n\geq 0}$. Show that $(X_n^{(i)})_{n\geq 0}$ is a Markov chain on R_i^+. Show that there is a unique stationaty probability measure π_i for the chain $(X_n^{(i)})_{n\geq 0}$.

(b) Show that if π is a stationary probability measure for $(X_n)_{n\geq 0}$, then there exists $\alpha_1, \alpha_2, \ldots \in [0, 1]$ with $\sum_{i\in\mathbb{N}} \alpha_i = 1$, such that $\pi = \sum_{i\in\mathbb{N}} \alpha_i \pi_i$.

Exercise 12.3 We consider the following modification of the *Ehrenfest* model of diffusion: N molecules are distributed in containers (I) and (II). At each step, we choose at random one molecule and place it

- into (I) with a probability $\frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}$,
- into (II) with a probability $\frac{1}{1+e^{-\beta\varepsilon}}$,

where $\beta > 0$ is the inverse temperature and $\varepsilon > 0$ an energy penalty.

(a) Let X_n denote the number of molecules in (I) at step n (starting from some initial distribution $X_0 \sim \mu$). Show that $(X_n)_{n\geq 0}$ is a Markov chain and find its transition probability.

(b) Show that the binomial distribution with parameters N and $\frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}$ is a reversible probability measure for $(X_n)_{n\geq 0}$.

(c) How many molecules will be on average in (I) as $n \to \infty$? What is the high temperature and low temperature behaviour of this quantity?

Submission deadline: 13:15, May 23.

Location: During exercise class or in the tray outside of HG E 65.

Class assignment:

Students	Time & Date	Room	Assistant
A-K	Thu 09-10	HG D 7.2	Maximilian Nitzschner
L-Z	Thu 12-13	HG D 7.2	Daniel Contreras

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2019/fs/401-3602-00L/