ETH Zürich, FS 2019
D-MATH
Prof. Dr. Vincent Tassion

Coordinator
Daniel Contreras Salinas

Applied Stochastic Processes

Exercise sheet 13

In the following exercises we consider $(X_n)_{n\geq 0}$ a Markov chain with transition probability p on a countable state space E.

Exercise 13.1 Assume that p is irreducible and aperiodic. Prove that for all $x, y \in E$ there exists n_0 such that for all $n \geq n_0$ we have $p^{(n)}_{x,y} > 0$.

Exercise 13.2 Lazy Markov chain
Let us define for all $x, y \in E$,

$$q_{x,y} = \frac{1}{2}\delta_{x,y} + \frac{1}{2}p_{x,y}.$$

(a) Prove that q is a transition probability. We define $(\widetilde{X}_n)_{n\geq 0}$ the Markov chain with this transition probability, which is called the *lazy* version of $(X_n)_{n\geq 0}$.

(b) Assume that p is irreducible. Prove that q is irreducible and aperiodic.

(c) Assume that $(X_n)_{n\geq 0}$ is positive recurrent. Prove that $(\widetilde{X}_n)_{n\geq 0}$ is positive recurrent. What is the stationary distribution for $(\widetilde{X}_n)_{n\geq 0}$?

Exercise 13.3 Assume that p is irreducible.

(a) Show that there is an integer $d \geq 1$ and a partition

$$E = C_0 \cup C_1 \cup \cdots \cup C_{d-1}$$

such that (setting $C_{nd+r} = C_r$)

(i) $p^{(n)}_{x,y} > 0$ only if $x \in C_r$ and $y \in C_{r+n}$ for some $r \in \{0, \ldots, d-1\}$;

(ii) $p^{(nd)}_{x,y} > 0$ for all sufficiently large n, for all $x, y \in C_r$, for all $r \in \{0, \ldots, d-1\}$.

(b) Let λ be a probability measure on E with $\sum_{x\in C_0} \lambda_x = 1$. Show that for $r \in \{0, 1, ..., d-1\}$ and $y \in C_r$, one has

$$\mathbf{P}_\lambda[X_{nd+r} = y] \to \frac{d}{\mathbf{E}_y[H_y^+]}, \qquad \text{as } n \to \infty.$$

Hint: *Set $Y_n = X_{nd+r}$ and conclude from (a) that the transition probability associated to $(Y_n)_{n\geq 0}$ is irreducible and aperiodic.*

Submission deadline: 13:15, May 31.

Location: In the tray outside of HG E 65.

Class assignment:

Students	Time & Date	Room	Assistant
A-K	Thu 09-10	HG D 7.2	Maximilian Nitzschner
L-Z	Thu 12-13	HG D 7.2	Daniel Contreras

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2019/fs/401-3602-00L/