Applied Stochastic Processes

Exercise sheet 13

In the following exercises we consider \( (X_n)_{n \geq 0} \) a Markov chain with transition probability \( p \) on a countable state space \( E \).

Exercise 13.1 Assume that \( p \) is irreducible and aperiodic. Prove that for all \( x, y \in E \) there exists \( n_0 \) such that for all \( n \geq n_0 \) we have \( p^{(n)}_{x,y} > 0 \).

Exercise 13.2 Lazy Markov chain
Let us define for all \( x, y \in E \),
\[
q_{x,y} = \frac{1}{2} \delta_{x,y} + \frac{1}{2} p_{x,y}.
\]
(a) Prove that \( q \) is a transition probability. We define \( (\tilde{X}_n)_{n \geq 0} \) the Markov chain with this transition probability, which is called the lazy version of \( (X_n)_{n \geq 0} \).
(b) Assume that \( p \) is irreducible. Prove that \( q \) is irreducible and aperiodic.
(c) Assume that \( (X_n)_{n \geq 0} \) is positive recurrent. Prove that \( (\tilde{X}_n)_{n \geq 0} \) is positive recurrent. What is the stationary distribution for \( (\tilde{X}_n)_{n \geq 0} \)?

Exercise 13.3 Assume that \( p \) is irreducible.
(a) Show that there is an integer \( d \geq 1 \) and a partition
\[
E = C_0 \cup C_1 \cup \cdots \cup C_{d-1}
\]
such that (setting \( C_{nd+r} = C_r \))
(i) \( p^{(n)}_{x,y} > 0 \) only if \( x \in C_r \) and \( y \in C_{r+n} \) for some \( r \in \{0, \ldots, d-1\} \);
(ii) \( p^{(nd)}_{x,y} > 0 \) for all sufficiently large \( n \), for all \( x, y \in C_r \), for all \( r \in \{0, \ldots, d-1\} \).
(b) Let \( \lambda \) be a probability measure on \( E \) with \( \sum_{x \in C_0} \lambda_x = 1 \). Show that for \( r \in \{0,1,\ldots,d-1\} \) and \( y \in C_r \), one has
\[
\mathbb{P}_{\lambda}[X_{nd+r} = y] \to \frac{d}{\mathbb{E}_y[H^+_y]}, \quad \text{as } n \to \infty.
\]
Hint: Set \( Y_n = X_{nd+r} \) and conclude from (a) that the transition probability associated to \( (Y_n)_{n \geq 0} \) is irreducible and aperiodic.

Location: In the tray outside of HG E 65.
Class assignment:
Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2019/fs/401-3602-00L/