Coordinator
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Applied Stochastic Processes

Exercise sheet 2

Exercise 2.1 The Waiting Time Paradox.

(a) Let $(N_t)_{t\geq 0}$ be a standard Poisson process with rate $\lambda > 0$. Let $(S_n)_{n\in\mathbb{N}}$ be the arrival times for this process. For a fixed $t > 0$, let $A_t = t - S_{N_t}$ be the time passed after the most recent arrival (or after 0) in the process, and let $B_t = S_{N_t+1} - t$ be the time forward to the next arrival. Let $T_1 \sim \text{Exp}(\lambda)$. Show that A_t and B_t are independent, that B_t is distributed as T_1 and that A_t is distributed as $T_1 \wedge t$.

(b) Let $L_t = A_t + B_t = S_{N_t+1} - S_{N_t}$ be the length of the interarrival interval covering t. Show that L_t has density

$$f_t(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & \text{if } 0 < x < t, \\ \lambda(1+\lambda t)e^{-\lambda x} & \text{if } x \geq t. \end{cases}$$

Show that $\mathrm{E}[L_t]$ converges to $2\mathrm{E}[T_1]$ as $t \to \infty$. Since L_t is the time between two consecutive arrivals, we would expect $\mathrm{E}[L_t] = \mathrm{E}[T_1]$. Give an intuitive resolution of the apparent paradox.

Exercise 2.2 Compound Poisson process.
Let $(N_t)_{t\geq 0}$ be a standard Poisson process with rate $\lambda > 0$ and $(X_k)_{k\in\mathbb{N}}$ a sequence of real-valued i.i.d. random variables with common distribution μ such that $(N_t)_{t\geq 0}$ and $(X_k)_{k\in\mathbb{N}}$ are independent. Define the process $Z = (Z_t)_{t\geq 0}$ by

$$Z_t := \sum_{k=1}^{N_t} X_k, \quad t \geq 0.$$

Z is called a *compound Poisson process* with rate λ and *jump size distribution* μ.

(a) For $t > 0$ determine the characteristic function of Z_t.

(b) Prove that Z has stationary and independent increments.

(c) Show that if $\mathrm{P}[X_i = 1] = 1 - \mathrm{P}[X_i = 0] = p$, then Z is a Poisson process with rate λp.

Exercise 2.3 Let $(N_t)_{t\geq 0}$ be a standard Poisson process with rate $\lambda > 0$. For every $n \in \mathbb{N}$ let $(X_i^{(n)})_{i\in\mathbb{N}_0}$ be a sequence of i.i.d. random variables with distribution Bernoulli(λ/n). Define

$$N_t^{(n)} = \sum_{i=0}^{\lfloor nt \rfloor} X_i^{(n)}, \quad t \geq 0.$$

Show that for all k, for all $0 \leq t_1 < \cdots < t_k < \infty$ and for all $f : \mathbb{N}^k \to \mathbb{R}$ bounded, we have that

$$\mathrm{E}[f(N_{t_1}^{(n)}, \ldots, N_{t_k}^{(n)})] \xrightarrow[n\to\infty]{} \mathrm{E}[f(N_{t_1}, \ldots, N_{t_k})]. \tag{1}$$

Hint: *Prove that for all* $(i_1, \ldots, i_k) \in \mathbb{N}^k$

$$\mathrm{P}[N_{t_1}^{(n)} = i_1, \ldots, N_{t_k}^{(n)} = i_k] \xrightarrow[n\to\infty]{} \mathrm{P}[N_{t_1} = i_1, \ldots, N_{t_k} = i_k]$$

and that this implies (1).

Submission deadline: 13:15, Mar. 7.

Location: During exercise class or in the tray outside of HG E 65.

Class assignment:

Students	Time & Date	Room	Assistant
A-K	Thu 09-10	HG D 7.2	Maximilian Nitzschner
L-Z	Thu 12-13	HG D 7.2	Daniel Contreras

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2019/fs/401-3602-00L/