Coordinator
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Applied Stochastic Processes

Exercise sheet 3

Exercise 3.1 Largest gap in a Poisson process.
Let $(N_t)_{t\geq 0}$ be a homogeneous Poisson process with parameter $\lambda > 0$. The largest gap up to time t is defined as

$$L_t = \max_{k\geq 1}(S_k \wedge t - S_{k-1} \wedge t).$$

In this exercise we are going to show that P-almost surely

$$\limsup_{t\to\infty} \frac{L_t}{\log t} \leq \lambda^{-1}.$$

(a) Let $\varepsilon > 0$. Use Borel-Cantelli's lemma to show that P-almost surely

$$\max_{1\leq k\leq n} T_k \leq \frac{1+\varepsilon}{\lambda} \log(n/\lambda)$$

for n large enough, where the T_k denote the inter-arrival times of the process.

(b) Show that P-almost surely

$$N_t + 1 \leq (1+\varepsilon)t\lambda$$

for large t enough.

(c) Conclude that P-almost surely $\limsup_{t\to\infty} \frac{L_t}{\log t} \leq \lambda^{-1}$.

Exercise 3.2 Let $(N_t)_{t\geq 0}$ be a homogeneous Poisson process with rate $\lambda > 0$. Let us define for every $t \geq 0$, $\widetilde{N}_t := N_t + \lfloor t \rfloor$, where $\lfloor\cdot\rfloor$ is the integer part function.

(a) Show that $(\widetilde{N}_t)_{t\geq 0}$ has independent increments and that for all $t \geq 0$ fixed

$$\mathrm{P}[\widetilde{N}_{t+h} - \widetilde{N}_t = 1] = \lambda h + o(h) \text{ and } \mathrm{P}[\widetilde{N}_{t+h} - \widetilde{N}_t \geq 2] = o(h) \text{ as } h \to 0.$$

(b) Show that $(\widetilde{N}_t)_{t\geq 0}$ is not an inhomogeneous Poisson process. Explain why this is not a contradiction with the infinitesimal characterization of inhomogeneous Poisson processes.

Exercise 3.3 Let $(N_t)_{t\geqslant 0}$ be an inhomogeneous Poisson process with $N_0 = 0$ and rate $\rho(t) = \alpha t$, where α is a positive constant. Let

$$S_n := \inf\{t > 0 \ : \ N_t = n\}, \quad n = 1, 2, \ldots$$

(a) Prove that for every $k \geq 1$, $(S_1, \ldots, S_k)$ and $(R^{-1}(\widetilde{T}_1), \ldots, R^{-1}(\widetilde{T}_1 + \cdots + \widetilde{T}_k))$ have the same distribution, where $(\widetilde{T}_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence of random variables with distribution Exp(1), and

$$R(t) = \int_0^t \rho(s)ds \text{ for every } t \geq 0.$$

(b) Calculate the explicit joint distribution of $(S_1, S_2 - S_1, \ldots, S_k - S_{k-1})$ for every $k \geq 2$. Conclude that the inter-arrival times of the process $(N_t)_{t\geq 0}$ are not independent.

Submission deadline: 13:15, Mar. 14.

Location: During exercise class or in the tray outside of HG E 65.

Class assignment:

Students	Time & Date	Room	Assistant
A-K	Thu 09-10	HG D 7.2	Maximilian Nitzschner
L-Z	Thu 12-13	HG D 7.2	Daniel Contreras

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2019/fs/401-3602-00L/