Exercise 3.1 Largest gap in a Poisson process.
Let \((N_t)_{t \geq 0}\) be a homogeneous Poisson process with parameter \(\lambda > 0\). The largest gap up to time \(t\) is defined as
\[
L_t = \max_{k \geq 1} (S_k \wedge t - S_{k-1} \wedge t).
\]
In this exercise we are going to show that \(P\)-almost surely
\[
\limsup_{t \to \infty} \frac{L_t}{\log t} \leq \lambda - 1.
\]
(a) Let \(\varepsilon > 0\). Use Borel-Cantelli’s lemma to show that \(P\)-almost surely
\[
\max_{1 \leq k \leq n} T_k \leq \frac{1 + \varepsilon}{\lambda} \log(n/\lambda)
\]
for \(n\) large enough, where the \(T_k\) denote the inter-arrival times of the process.
(b) Show that \(P\)-almost surely
\[
N_t + 1 \leq (1 + \varepsilon)t\lambda
\]
for large \(t\) enough.
(c) Conclude that \(P\)-almost surely \(\limsup_{t \to \infty} \frac{L_t}{\log t} \leq \lambda - 1\).

Exercise 3.2 Let \((N_t)_{t \geq 0}\) be a homogeneous Poisson process with rate \(\lambda > 0\). Let us define for every \(t \geq 0\), \(\tilde{N}_t := N_t + \lfloor t \rfloor\), where \(\lfloor \cdot \rfloor\) is the integer part function.
(a) Show that \((\tilde{N}_t)_{t \geq 0}\) has independent increments and that for all \(t \geq 0\) fixed
\[
P[\tilde{N}_{t+h} - \tilde{N}_t = 1] = \lambda h + o(h) \text{ and } P[\tilde{N}_{t+h} - \tilde{N}_t \geq 2] = o(h) \text{ as } h \to 0.
\]
(b) Show that \((\tilde{N}_t)_{t \geq 0}\) is not an inhomogeneous Poisson process. Explain why this is not a contradiction with the infinitesimal characterization of inhomogeneous Poisson processes.

Exercise 3.3 Let \((N_t)_{t \geq 0}\) be an inhomogeneous Poisson process with \(N_0 = 0\) and rate \(\rho(t) = \alpha t\), where \(\alpha\) is a positive constant. Let
\[
S_n := \inf\{t > 0 : N_t = n\}, \quad n = 1, 2, \ldots
\]
(a) Prove that for every \(k \geq 1\), \((S_1, \ldots, S_k)\) and \(\left(R^{-1}(T_1), \ldots, R^{-1}(T_1 + \cdots + T_k)\right)\) have the same distribution, where \((T_i)_{i \in \mathbb{N}}\) is an i.i.d. sequence of random variables with distribution \(\text{Exp}(1)\), and
\[
R(t) = \int_0^t \rho(s)ds \text{ for every } t \geq 0.
\]
(b) Calculate the explicit joint distribution of \((S_1, S_2 - S_1, \ldots, S_k - S_{k-1})\) for every \(k \geq 2\). Conclude that the inter-arrival times of the process \((N_t)_{t \geq 0}\) are not independent.

Location: During exercise class or in the tray outside of HG E 65.

Class assignment:

<table>
<thead>
<tr>
<th>Students</th>
<th>Time &amp; Date</th>
<th>Room</th>
<th>Assistant</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-K</td>
<td>Thu 09-10</td>
<td>HG D 7.2</td>
<td>Maximilian Nitzschner</td>
</tr>
<tr>
<td>L-Z</td>
<td>Thu 12-13</td>
<td>HG D 7.2</td>
<td>Daniel Contreras</td>
</tr>
</tbody>
</table>

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Exercise sheets and further information are also available on: http://metaphor.ethz.ch/x/2019/fs/401-3602-00L/