ETH Zürich, FS 2019
D-MATH
Prof. Dr. Vincent Tassion

Coordinator
Daniel Contreras Salinas

Applied Stochastic Processes

Exercise sheet 4

Exercise 4.1 Campbell's formula
Let N be a point process on $(E, \mathcal{E})$ with intensity measure μ, where μ is s-finite. Let $u : E \to \mathbb{R}$ be a measurable function. Show that $\int u(x)N(dx)$ is a well defined random variable and that if we have $u \geq 0$ or $\int |u(x)|\mu(dx) < \infty$, then

$$\mathrm{E}\left[\int u(x)N(dx)\right] = \int u(x)\mu(dx).$$

Exercise 4.2 Let N be a Poisson point process on $\mathbb{R}$ with intensity measure $\mu = \lambda \cdot \mathrm{Leb}(\mathbb{R})$, where $\lambda > 0$ and $\mathrm{Leb}(\mathbb{R})$ is the Lebesgue measure on $\mathbb{R}$. Let us order the points in $(0, \infty)$ as $0 < X_1 < X_2 < \cdots$.

(a) Show that $(X_n)_{n\geq 1}$ are well defined random variables.

(b) Prove that the random variables

$$Y_1 = X_1, \quad Y_n = X_n - X_{n-1} \text{ for } n \geq 2$$

are i.i.d. with distribution $\mathrm{Exp}(\lambda)$.

Exercise 4.3

(a) (Mapping Theorem) Let N be a Poisson process with s-finite intensity measure μ on a state space E with corresponding σ-algebra $\mathcal{E}$. Let $f : (E, \mathcal{E}) \to (F, \mathcal{F})$ be a measurable function and let $\mu^* = \mu \circ f^{-1}$ be the induced measure on F. Show that $N^* = N \circ f^{-1}$ is a Poisson process on F having intensity measure μ^*.

(b) Let N be a Poisson point process on $\mathbb{R}^d$ with intensity measure $\mu = \lambda \cdot \mathrm{Leb}(R^d)$, where $\lambda > 0$. Let B_r the ball of radius r around the origin. Prove that a.s.

$$\lim_{r\to\infty} \frac{N(B_r)}{|B_r|} = \lambda$$

where $|B_r|$ is the volume of B_r.
Hint: *Use the Mapping Theorem to study the process* $(N(B_r))_{r>0}$.

Submission deadline: 13:15, Mar. 21.

Location: During exercise class or in the tray outside of HG E 65.

Class assignment:

Students	Time & Date	Room	Assistant
A-K	Thu 09-10	HG D 7.2	Maximilian Nitzschner
L-Z	Thu 12-13	HG D 7.2	Daniel Contreras

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2019/fs/401-3602-00L/