Applied Stochastic Processes

Exercise sheet 5

Exercise 5.1 Set $E := [0, 1]$. We say that a set is co-countable if its complement is countable. Let $\mathcal{E}$ be the family of subsets of $E$ that are either countable or co-countable.

(a) Show that $\mathcal{E}$ is a $\sigma$-algebra.

(b) Find a measure $\eta$ on $(E, \mathcal{E})$ such that for all $B \in \mathcal{E}$, $\eta(B) \in \{0, 1\}$, which is not of the form $\delta_x$ for some $x \in E$.

(c) Show that there exists a point process on $(E, \mathcal{E})$ which is not proper.

Exercise 5.2 Let $N$ be a point process on $(E, \mathcal{E})$ with intensity measure $\mu$ and let $B \in \mathcal{E}$. Let $\mathcal{L}_N$ be the Laplace functional of $N$, which is given by

$$\mathcal{L}_N(u) = E \left[ \exp \left( - \int_E u(x) N(dx) \right) \right].$$

for all $u : E \to \mathbb{R}_+$ measurable.

(a) Show that if $\mu(B) < \infty$, then

$$\mu(B) = -\frac{d}{dt} \mathcal{L}_N(t1_B) \bigg|_{t=0}.$$

(b) We no longer assume that $\mu(B) < \infty$. Show that

$$P[N(B) = 0] = \lim_{t \to \infty} \mathcal{L}_N(t1_B).$$

Exercise 5.3 Poisson Boolean model

Let $N = \sum_i \delta_{X_i}$ be a Poisson point process on $\mathbb{R}^d$ with intensity measure $\mu = \text{Leb}(\mathbb{R}^d)$. Let us consider $(R_i)_i$, a sequence of i.i.d. positive random variables with law $\rho$, and independent of $N$. We define the occupied set by $\mathcal{O} = \bigcup_i B(X_i, R_i)$, where $B(x, r) \subset \mathbb{R}^d$ is the closed ball of center $x$ and radius $r$.

(a) Let $N_0$ the number of balls $B(X_i, R_i)$ which contain the origin of $\mathbb{R}^d$. Show that $N_0$ is a well defined random variable with distribution Poisson $\left( \int_{\mathbb{R}^d} \int_0^\infty \rho(dr) \mu(dx) \right)$.

(b) Show that the event $\{\mathcal{O} = \mathbb{R}^d\}$ is measurable and that $P[\mathcal{O} = \mathbb{R}^d] = 1$ if and only if $\int_0^\infty r^d \rho(dr) = \infty$. 
Submission deadline: 13:15, Mar. 28.

Location: During exercise class or in the tray outside of HG E 65.

Class assignment:

<table>
<thead>
<tr>
<th>Students</th>
<th>Time &amp; Date</th>
<th>Room</th>
<th>Assistant</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-K</td>
<td>Thu 09-10</td>
<td>HG D 7.2</td>
<td>Maximilian Nitzschner</td>
</tr>
<tr>
<td>L-Z</td>
<td>Thu 12-13</td>
<td>HG D 7.2</td>
<td>Daniel Contreras</td>
</tr>
</tbody>
</table>

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2019/fs/401-3602-00L/