

Applied Stochastic Processes

Exercise sheet 5

Exercise 5.1 Set $E := [0, 1]$. We say that a set is *co-countable* if its complement is countable. Let \mathcal{E} be the family of subsets of E that are either countable or co-countable.

- Show that \mathcal{E} is a σ -algebra.
- Find a measure η on (E, \mathcal{E}) such that for all $B \in \mathcal{E}$, $\eta(B) \in \{0, 1\}$, which is not of the form δ_x for some $x \in E$.
- Show that there exists a point process on (E, \mathcal{E}) which is not proper.

Exercise 5.2 Let N be a point process on (E, \mathcal{E}) with intensity measure μ and let $B \in \mathcal{E}$. Let \mathcal{L}_N be the Laplace functional of N , which is given by

$$\mathcal{L}_N(u) = \mathbb{E} \left[\exp \left(- \int_E u(x) N(dx) \right) \right].$$

for all $u : E \rightarrow \mathbb{R}_+$ measurable.

- Show that if $\mu(B) < \infty$, then

$$\mu(B) = - \frac{d}{dt} \mathcal{L}_N(t1_B) \Big|_{t=0}.$$

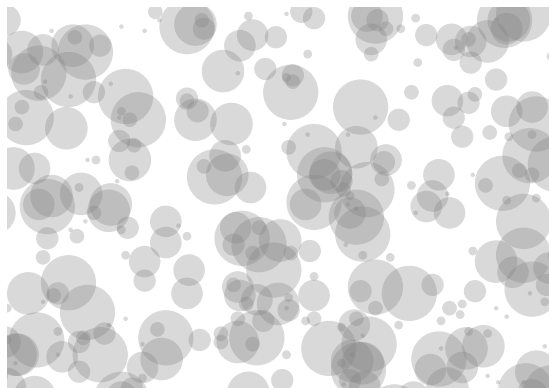
- We no longer assume that $\mu(B) < \infty$. Show that

$$\mathbb{P}[N(B) = 0] = \lim_{t \rightarrow \infty} \mathcal{L}_N(t1_B).$$

Exercise 5.3 Poisson Boolean model

Let $N = \sum_i \delta_{X_i}$ be a Poisson point process on \mathbb{R}^d with intensity measure $\mu = \text{Leb}(\mathbb{R}^d)$. Let us consider $(R_i)_i$ a sequence of i.i.d. positive random variables with law ρ , and independent of N . We define the *occupied* set by $\mathcal{O} = \bigcup_i B(X_i, R_i)$, where $B(x, r) \subset \mathbb{R}^d$ is the closed ball of center x and radius r .

- Let N_0 the number of balls $B(X_i, R_i)$ which contain the origin of \mathbb{R}^d . Show that N_0 is a well defined random variable with distribution Poisson $\left(\int_{\mathbb{R}^d} \int_{|x|}^{\infty} \rho(dr) \mu(dx) \right)$.
- Show that the event $\{\mathcal{O} = \mathbb{R}^d\}$ is measurable and that $\mathbb{P}[\mathcal{O} = \mathbb{R}^d] = 1$ if and only if $\int_0^{\infty} r^d \rho(dr) = \infty$.



Submission deadline: 13:15, Mar. 28.

Location: During exercise class or in the tray outside of HG E 65.

Class assignment:

Students	Time & Date	Room	Assistant
A-K	Thu 09-10	HG D 7.2	Maximilian Nitzschner
L-Z	Thu 12-13	HG D 7.2	Daniel Contreras

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Exercise sheets and further information are also available on:
<http://metaphor.ethz.ch/x/2019/fs/401-3602-00L/>