Applied Stochastic Processes

Exercise sheet 6

Exercise 6.1  Show that a renewal process with renewal function $M(t) = ct$, $t \geq 0$ for some constant $c > 0$ is a Poisson process. 

Hint: The Laplace transform determines the distribution.

Exercise 6.2  Vehicles of random lengths arrive at a gate. Let $L_k$ denote the length of the $k$-th vehicle. We assume that the random variables $L_k$ are i.i.d. with distribution $3 + \text{Geometric}(1/2)$. The first vehicle that arrives parks directly at the gate. The vehicles arriving afterwards queue behind, leaving a random distance to the vehicle parked in front of themselves. We assume that these distances are independent and uniformly distributed on $[0, 1]$.

(a) For $x \geq 0$, let $N_x$ denote the number of vehicles parked at distance at most $x$ from the gate.
Compute $\lim_{x \to \infty} N_x/x$.

(b) Suppose that the $k$-th vehicle is carrying $D_k$ people, where the random variable $D_k$ is distributed as $1 + \text{Binomial}(2L_k, 1/2)$. For $x \geq 0$ let $\tilde{N}_x$ denote the number of people inside the vehicles parked at distance at most $x$ from the gate. Estimate $\tilde{N}_x$ for $x$ large enough.

Exercise 6.3  Central Limit Theorem for Renewal Processes
If $(N_i)_{i \geq 0}$ is a renewal process with inter-arrival times $T_i$, $i \geq 1$, not a.s. constant and such that $E[T_i^2] < \infty$, show that when $t \to \infty$,

$$Z_t := \frac{N_t - t/\mu}{\sigma(t/\mu)^{\frac{3}{2}}}$$

converges in law to the standard normal distribution, where $\mu = E[T_1]$ and $\sigma^2 = \text{Var}(T_1) > 0$.

Hint: Let $S_n := T_1 + \ldots + T_n$, then by the central limit theorem

$$\lim_{n \to \infty} P[(S_n - n\mu)/\sigma \sqrt{n} \leq x] = \Phi(x)$$

uniformly in $x \in \mathbb{R}$, where $\Phi$ denotes the distribution function of the standard normal distribution.

Submission deadline: 13:15, Apr. 4.

Location: During exercise class or in the tray outside of HG E 65.

Class assignment:

<table>
<thead>
<tr>
<th>Students</th>
<th>Time &amp; Date</th>
<th>Room</th>
<th>Assistant</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-K</td>
<td>Thu 09-10</td>
<td>HG D 7.2</td>
<td>Maximilian Nitzschner</td>
</tr>
<tr>
<td>L-Z</td>
<td>Thu 12-13</td>
<td>HG D 7.2</td>
<td>Daniel Contreras Salinas</td>
</tr>
</tbody>
</table>

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Exercise sheets and further information are also available on: http://metaphor.ethz.ch/x/2019/fs/401-3602-00L/