Applied Stochastic Processes

Exercise sheet 7

Exercise 7.1  Cycles of operation and repair of a machine.
Let \((U_i, V_i)_{i \in \mathbb{N}}\) be a sequence of i.i.d. random vectors with \(U_i \geq 0, V_i \geq 0\). Assume that \(T_i = U_i + V_i\) is not almost surely equal to 0 and denote by \(F\) its distribution function. We interpret \(U_i\) and \(V_i\) as alternating periods when a given machine is operational or in repair. The period \(U_1\) begins at time 0. For \(t \geq 0\) we define \(Y_t = 1\) if the machine is operational at time \(t\) and \(Y_t = 0\) otherwise.

Let \(g(t) = P[Y_t = 1]\) denote the probability of the machine being operational at time \(t \geq 0\), and \(g(t) = 0\) for \(t < 0\). We also define \(h(t) = P[U_1 > t]\).

(a) Prove that for \(t \geq 0\)

\[ g(t) = h(t) + \int_0^t g(t-s) dF(s), \]

i.e. that \(g\) is the solution of the \((h, F)\)-renewal equation.
(b) Assume that \(E[U_1] < \infty\) and that \(F\) is non-arithmetic. Show that the function \(h\) is directly Riemann integrable and conclude that

\[ \lim_{t \to \infty} g(t) = \frac{E[U_1]}{E[U_1] + E[V_1]}. \]

Exercise 7.2  Let \(N\) be a renewal process with renewal times \((S_k)_{k \geq 0}\), where \(S_0 = 0\), and interarrival distribution \(F\) having finite mean \(\mu > 0\). Denote by \(A\) the age process of \(N\), i.e. \(A_t = t - S_{N_t}\) for \(t \geq 0\). For \(x \geq 0\), set \(\varphi_x(t) = P[A_t \leq x]\) for \(t \geq 0\), and \(\varphi_x(t) = 0\) for \(t < 0\).

(a) Show that \(\varphi_x\) satisfies the renewal equation

\[ \varphi_x(t) = 1_{\{t \leq x\}}(1 - F(t)) + \int_0^t \varphi_x(t-s) dF(s) \text{ for } t \geq 0. \]

(b) Assume that \(F\) is non-arithmetic. Compute \(\lim_{t \to \infty} \varphi_x(t)\). Deduce that \(A_t\) converges in distribution to some random variable \(A_\infty\) as \(t \to \infty\).

Submission deadline: 13:15, Apr. 11.

Location: During exercise class or in the tray outside of HG E 65.

Class assignment:

<table>
<thead>
<tr>
<th>Students</th>
<th>Time &amp; Date</th>
<th>Room</th>
<th>Assistant</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-K</td>
<td>Thu 09-10</td>
<td>HG D 7.2</td>
<td>Maximilian Nitzschner</td>
</tr>
<tr>
<td>L-Z</td>
<td>Thu 12-13</td>
<td>HG D 7.2</td>
<td>Daniel Contreras Salinas</td>
</tr>
</tbody>
</table>

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2019/fs/401-3602-00L/