ETH Zürich, FS 2019
D-MATH
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Coordinator
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Applied Stochastic Processes

Exercise sheet 9

Exercise 9.1 Let $(X_n)_{n\geq 0}$ be a homogeneous Markov chain with countable state space E and transition probabilities $(p_{x,y})_{x,y\in E}$. Let $C \subseteq E$ such that $E\backslash C$ is finite. Define $p_{x,C}(n) = \sum_{y\in C} p_{x,y}(n)$. Suppose that for each $x \in E\backslash C$ there exists an $n(x)$ such that $p_{x,C}(n(x)) > 0$. Let $\tau_C = \inf\{n \geq 0 \ : \ X_n \in C\}$, $\varepsilon = \min\{p_{x,C}(n(x)) \ : \ x \in E\backslash C\}$, and $N = \max\{n(x) \ : \ x \in E\backslash C\}$. Show that for all $k \in \mathbb{N}$,

$$\mathbf{P}_x[\tau_C > kN] \leq (1-\varepsilon)^k \quad \forall x \in E.$$

Exercise 9.2 Let $(X_n)_{n\geq 0}$ be a Markov chain with state space $E = \{0, 1, \ldots, N\}$. Let us fix $0 \leq x \leq N$ and suppose that under $\mathbf{P}_x$, X_n is a martingale for the canonical filtration $\mathcal{F}_n$. We define $\tau_y = \inf\{n \geq 0; X_n = y\}$. Suppose that $\mathbf{P}_x[\tau_0 \wedge \tau_N < \infty] > 0$.

(a) Show that 0 and N are *absorbing states*, i.e., $p_{0,0} = p_{N,N} = 1$.

(b) Show that $\mathbf{P}_x[\tau_N < \tau_0] = \frac{x}{N}$.

(c) Consider the Gambler ruin chain. Assume that the gambler starts with $k > 0$ dollars. What is the probability that he/she finishes with 0 dollar?

Exercise 9.3 Wright–Fisher model.
Let us consider the following inheritance model for a particular gene with two alleles A and a. In each generation there are m individuals, each one having 2 alleles of the same gene. Each individual of generation $n+1$ chooses its alleles independently from the other individuals and uniformly between the $2m$ possible alleles of generation n. Let us suppose that there are $k \in \{0, \ldots, 2m\}$ alleles of type A in the generation 0. Let X_n be the number of alleles of type A in generation n.

(a) Prove that $(X_n)_{n\geq 0}$ is a Markov chain and find its transition probability $(p_{i,j})_{0\leq i,j\leq 2m}$.

(b) Show that the probability that the allele a disappears before allele A in some genration is $\frac{k}{2m}$.

Submission deadline: 13:15, May 2.

Location: During exercise class or in the tray outside of HG E 65.

Class assignment:

Students	Time & Date	Room	Assistant
A-K	Thu 09-10	HG D 7.2	Maximilian Nitzschner
L-Z	Thu 12-13	HG D 7.2	Daniel Contreras

Office hours (Präsenz): Mon. and Thu., 12:00-13:00 in HG G 32.6.

Exercise sheets and further information are also available on:
http://metaphor.ethz.ch/x/2019/fs/401-3602-00L/