Applied Stochastic Processes

Exercise sheet 1

Exercise 1.1
(a) Let $T_1, \ldots, T_k$ be i.i.d. random variables with $T_1 \sim \text{Exp}(\lambda)$ and $S_k = \sum_{i=1}^{k} T_i$. Show that $S_k \sim \text{Gamma}(k, \lambda)$.
(b) A real and positive random variable $X$ is said to have the memoryless property if $P[X \geq x] > 0$ for all $x > 0$ and $P[X \geq x + y \mid X \geq x] = P[X \geq y]$ for all $x, y > 0$.
Prove that a continuous and positive random variable $X$ has the memoryless property if and only if $X \sim \text{Exp}(\lambda)$ for some $\lambda > 0$.

Exercise 1.2 Wald’s Equation.
Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of i.i.d. random variables with $E[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2 < \infty$ and $(S_n)_{n \in \mathbb{N}}$ the sequence of partial sums defined by $S_0 := 0$ and $S_n := \sum_{i=1}^{n} X_i$.
For a non-negative, integer-valued random variable $N$, which is independent of $(X_i)_{i \in \mathbb{N}}$, let $S_N$ denote the random sum defined by $S_N := \sum_{i=1}^{N} X_i$.
(a) Suppose that $E[N] < \infty$. Prove $E[S_N \mid N] = \mu N$ and $E[S_N] = \mu E[N]$.
Hint: Do not forget to argue that $S_N$ is integrable.
(b) Suppose that $E[N^2] < \infty$. Show that $E[S_N^2 \mid N] = \sigma^2 N + \mu^2 N^2$ and $\text{Var}(S_N) = \sigma^2 E[N] + \mu^2 \text{Var}(N)$.

Exercise 1.3 Spatial Poisson process.
Let countably many points be distributed in $\mathbb{R}^2$ according to the following rule:
1. For a bounded set $A \in \mathcal{B}(\mathbb{R}^2)$ the number of points $N(A)$ lying in the set $A$ is Poisson-distributed with parameter $\mu(A)$ where $\mu(A) := \lambda |A|$ with $\lambda > 0$ and $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^2$.
2. $N(A_1), \ldots, N(A_k)$ are independent for disjoint bounded sets $A_1, \ldots, A_k \in \mathcal{B}(\mathbb{R}^2)$.
(a) For fixed $r > 0$ we define $B_r := \{x \in \mathbb{R}^2 : \|x\| \leq r\}$ (where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^2$) and $D := \inf\{r > 0 : N(B_r) > 0\}$. Determine the distribution function and the density of $D$.
(b) Compute for $u > r$ the limit $\lim_{r \to 0} P[N(B_u) = 1 \mid N(B_r) = 1]$. 


Solution 1.1

(a) For a given random variable $X \sim \text{Gamma}(k, \lambda)$ we know that its density function is given by

$$f_X(t) = \lambda^k \frac{t^{k-1}}{(k-1)!} e^{-\lambda t} \quad \text{for } t > 0.$$ 

We prove by induction that this is the density of $S_k \forall k \in \mathbb{N}$. First note that $S_1 = T_1 \sim \text{Exp}(\lambda)$ has density $g(t) = \lambda e^{-\lambda t}$, so $S_1 \sim \text{Gamma}(1, \lambda)$. Now, suppose that $S_k \sim \text{Gamma}(k, \lambda)$ and calculate the density for $S_{k+1}$ (using independence of $T_{k+1}$ and $S_k$, and convolution):

$$g^{*(k+1)}(t) = g \ast g^k(t) = \int_0^t \lambda e^{-\lambda(t-s)} \lambda^{k+1} s^{k-1} (k-1)! e^{-\lambda s} ds
= \lambda^{k+1} e^{-\lambda t} \int_0^t \frac{s^{k-1}}{(k-1)!} ds = \lambda^{k+1} \frac{t^k}{k!} e^{-\lambda t}.$$ 

Hence $S_{k+1} \sim \text{Gamma}(k+1, \lambda)$ and the induction is complete.

Remarks:

- The Gamma($\nu, \lambda$) distribution is defined for general parameters $\nu, \lambda > 0$ and has density

$$g(t) = \lambda^{\nu} \frac{t^{\nu-1}}{\Gamma(\nu)} e^{-\lambda t}, \quad t > 0,$$

where $\Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t} dt$ is the gamma function.

- For $\nu = k \in \mathbb{N}$ this is also called the Erlang-$k$ distribution, which represents the law of the time spent up to the $k$-th arrival in a Poisson process with intensity $\lambda$.

- We can calculate the characteristic function of the Gamma($k, \lambda$) distribution via the characteristic function of Exp($\lambda$):

$$\varphi_{T_i}(u) = \mathbb{E}[e^{iuT_i}] = \int_0^\infty \lambda e^{-(\lambda-iu)t} dt = \frac{\lambda}{\lambda-iu}.$$ 

$$\varphi_{S_k}(u) = \mathbb{E}[e^{iu \sum_{j=1}^k T_j}] = \mathbb{E} \left[ \prod_{j=1}^k e^{iu T_j} \right] \overset{\text{iid}}{=} \mathbb{E} \left[ e^{iu T_1} \right]^k = \left( \frac{\lambda}{\lambda-iu} \right)^k.$$ 

(b) First, note that we can rewrite the memoryless condition as

$$P[X \geq x + y] = P[X \geq x]P[X \geq y] \quad \text{for any } x, y > 0.$$

Let us define for $x > 0$ the function $h(x) = P[X \geq x]$.

$\Leftarrow$ We know that if $X \sim \text{Exp}(\lambda)$, we have $h(x) = e^{-\lambda x}$. Then

$$h(x)h(y) = e^{-\lambda x} e^{-\lambda y} = e^{-\lambda(x+y)} = h(x+y).$$

$\Rightarrow$ Suppose that $X$ is a positive and continuous random variable. Then, $h$ is a continuous decreasing function which satisfies $h(x+y) = h(x)h(y)$ for all $x, y > 0$. Since $X$ is positive we know that $\lim_{x \to 0^+} h(x) = 1$. Also, since $X$ is not $\infty$ a.s., we have $\lim_{x \to \infty} h(x) = 0$. Thus, let us consider $x_0 > 0$ such that $1 > h(x_0) > 0$. It is easy to check that for any $n$ natural number $h(nx_0) = h(x_0)^n$ and $h(x_0/n) = h(x_0)^{1/n}$. This implies that $h(px_0/q) = h(x_0)^{p/q}$ for any $p, q$ natural numbers and thus $h(rx_0) = h(x_0)^r$ for every $r$ positive rational number. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $h$ is continuous, we conclude that $h(yx_0) = h(x_0)^y$ for all $y$ positive real number. Taking $x = yx_0$ and $\lambda = -\log(h(x_0))/x_0 > 0$ we conclude that $h(x) = e^{-\lambda x}$ for every $x > 0$. Therefore $X \sim \text{Exp}(\lambda)$. 


Solution 1.2

(a) For \( n \in \mathbb{N} \) set \( Y_n := \sum_{i=1}^{n} |X_i| \) and note that the \( |X_i| \) are i.i.d. and independent of \( N \). Hence, we have by the monotone convergence theorem and independence of the \( X_i \) and \( N \)

\[
E[|S_N|] = E\left[ \sum_{k=0}^{\infty} |S_k|1_{\{N=k\}} \right] \leq E\left[ \sum_{k=0}^{\infty} Y_k1_{\{N=k\}} \right]
\]

\[
= \sum_{k=0}^{\infty} E[Y_k1_{\{N=k\}}] = \sum_{k=0}^{\infty} E[Y_k]E[1_{\{N=k\}}]
\]

\[
= E[Y_1]\left( \sum_{k=0}^{\infty} kP[N = k] \right) = E[Y_1]E[N] < \infty.
\]

We have \( E[S_n] = \mu n \) for all \( n \in \mathbb{N} \). As \( N \) is independent of \((X_i)_{i\in\mathbb{N}}\), it follows for all \( n \in \mathbb{N}_0 \)

\[
E[S_{N1_{\{N=n\}}}] = E[S_n1_{\{N=n\}}] = E[S_n]E[1_{\{N=n\}}] = \mu n E[1_{\{N=n\}}] = E[\mu N 1_{\{N=n\}}].
\]

Hence, as \( N \) takes values in \( \mathbb{N}_0 \) so that \( \sigma(N) \) is generated by the sets \( \{N = n\} \), we have \( E[S_N | N] = \mu N \). This implies

\[
E[S_N] = E[E[S_N | N]] = E[\mu N] = \mu E[N].
\]

(b) We have \( E[S_n^2] = \sigma^2 n + \mu^2 n^2 \) for all \( n \in \mathbb{N} \). Since \((S_N)^2 \geq 0\), the conditional expectation \( E[(S_N)^2 | N] \) is well-defined. As \( N \) is independent of \((X_i)_{i\in\mathbb{N}}\), it follows for all \( n \in \mathbb{N} \)

\[
E[S_{N1_{\{N=n\}}}^2] = E[S_n^21_{\{N=n\}}]
\]

\[
= E[S_n^2]E[1_{\{N=n\}}]
\]

\[
= (\sigma^2 n + \mu^2 n^2)E[1_{\{N=n\}}]
\]

\[
= E[(\sigma^2 N + \mu^2 N^2)1_{\{N=n\}}].
\]

This implies \( E[S_N^2 | N] = \sigma^2 N + \mu^2 N^2 \). This leads to

\[
\text{Var}(S_N) = E[S_N^2] - E[S_N]^2
\]

\[
= E[E[S_N^2 | N]] - E[S_N]^2
\]

\[
= \sigma^2 E[N] + \mu^2 E[N^2] - \mu^2 E[N]^2
\]

\[
= \sigma^2 E[N] + \mu^2 \text{Var}(N).
\]

Solution 1.3

(a) For all \( r \geq 0 \):

\[
P[D > r] = P[N(B_r) = 0] = \exp(-\lambda \pi r^2).
\]

Hence, the distribution function \( F \) and the density function \( f \) of \( D \) are given by

\[
F(r) = (1 - \exp(-\lambda \pi r^2))1_{(r \geq 0)}, \quad f(r) = 2\lambda \pi r \exp(-\lambda \pi r^2)1_{(r \geq 0)}.
\]
(b) For all $u > r$:

\[
P[N(B_u) = 1 | N(B_r) = 1] = \frac{P[N(B_u \setminus B_r) = 0, N(B_r) = 1]}{P[N(B_r) = 1]}
\]

\[
= \frac{P[N(B_u \setminus B_r) = 0] P[N(B_r) = 1]}{P[N(B_r) = 1]}
\]

\[
= \exp(-\lambda \pi (u^2 - r^2))
\]

Hence, as $r \to 0$ we obtain

\[
P[N(B_u) = 1 | N(B_r) = 1] \to \exp(-\lambda \pi u^2).
\]