Applied Stochastic Processes

Exercise sheet 1

Exercise 1.1

(a) Let $T_1, \ldots, T_k$ be i.i.d. random variables with $T_1 \sim \text{Exp}(\lambda)$ and $S_k = \sum_{i=1}^{k} T_i$. Show that $S_k \sim \text{Gamma}(k, \lambda)$.

(b) A real and positive random variable X is said to have the *memoryless* property if $\text{P}[X \geq x] > 0$ for all $x > 0$ and

$$\text{P}[X \geq x + y \,|X \geq x] = \text{P}[X \geq y] \text{ for all } x, y > 0.$$

Prove that a continuous and positive random variable X has the memoryless property if and only if $X \sim \text{Exp}(\lambda)$ for some $\lambda > 0$.

Exercise 1.2 Wald's Equation.
Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables with $\text{E}[X_i] = \mu$ and $\text{Var}(X_i) = \sigma^2 < \infty$ and $(S_n)_{n\in\mathbb{N}}$ the sequence of partial sums defined by $S_0 := 0$ and $S_n := \sum_{i=1}^{n} X_i$.

For a non-negative, integer-valued random variable N, which is independent of $(X_i)_{i\in\mathbb{N}}$, let S_N denote the random sum defined by $S_N := \sum_{i=1}^{N} X_i$.

(a) Suppose that $\text{E}[N] < \infty$. Prove

$$\text{E}[S_N|N] = \mu N$$

and

$$\text{E}[S_N] = \mu\text{E}[N].$$

Hint: Do not forget to argue that S_N is integrable.

(b) Suppose that $\text{E}[N^2] < \infty$. Show that

$$\text{E}[S_N^2|N] = \sigma^2 N + \mu^2 N^2$$

and

$$\text{Var}(S_N) = \sigma^2\text{E}[N] + \mu^2\text{Var}(N).$$

Exercise 1.3 Spatial Poisson process.
Let countably many points be distributed in $\mathbb{R}^2$ according to the following rule:

1. For a bounded set $A \in \mathcal{B}(\mathbb{R}^2)$ the number of points $N(A)$ lying in the set A is Poisson-distributed with parameter $\mu(A)$ where $\mu(A) := \lambda|A|$ with $\lambda > 0$ and $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^2$.
2. $N(A_1), \ldots, N(A_k)$ are independent for disjoint bounded sets $A_1, \ldots, A_k \in \mathcal{B}(\mathbb{R}^2)$.

(a) For fixed $r > 0$ we define $B_r := \{x \in \mathbb{R}^2 : \|x\| \leq r\}$ (where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^2$) and $D := \inf\{r > 0 \,|\, N(B_r) > 0\}$. Determine the distribution function and the density of D.

(b) Compute for $u > r$ the limit $\lim_{r\to 0} \text{P}[N(B_u) = 1 \,|\, N(B_r) = 1]$.

Solution 1.1

(a) For a given random variable $X \sim \text{Gamma}(k, \lambda)$ we know that its density function is given by

$$f_X(t) = \lambda^k \frac{t^{k-1}}{(k-1)!} \mathrm{e}^{-\lambda t} \text{ for } t > 0.$$

We prove by induction that this is the density of S_k $\forall k \in \mathbb{N}$. First note that $S_1 = T_1 \sim \text{Exp}(\lambda)$ has density $g(t) = \lambda \mathrm{e}^{-\lambda t}$, so $S_1 \sim \text{Gamma}(1, \lambda)$. Now, suppose that $S_k \sim \text{Gamma}(k, \lambda)$ and calculate the density for S_{k+1} (using independence of T_{k+1} and S_k, and convolution):

$$\begin{aligned} g^{*(k+1)}(t) = g * g^{*k}(t) &= \int_0^t \lambda \mathrm{e}^{-\lambda(t-s)} \lambda^k \frac{s^{k-1}}{(k-1)!} \mathrm{e}^{-\lambda s} ds \\ &= \lambda^{k+1} \mathrm{e}^{-\lambda t} \int_0^t \frac{s^{k-1}}{(k-1)!} ds = \lambda^{k+1} \frac{t^k}{k!} \mathrm{e}^{-\lambda t}. \end{aligned}$$

Hence $S_{k+1} \sim \text{Gamma}(k+1, \lambda)$ and the induction is complete.

Remarks:

- The $\text{Gamma}(\nu, \lambda)$ distribution is defined for general parameters $\nu, \lambda > 0$ and has density

$$g(t) = \lambda^\nu \frac{t^{\nu-1}}{\Gamma(\nu)} \mathrm{e}^{-\lambda t}, \quad t > 0,$$

  where $\Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t} dt$ is the gamma function.
- For $\nu = k \in \mathbb{N}$ this is also called the Erlang-k distribution, which represents the law of the time spent up to the k-th arrival in a Poisson process with intensity λ.
- We can calculate the characteristic function of the $\text{Gamma}(k, \lambda)$ distribution via the characteristic function of $\text{Exp}(\lambda)$:

$$\varphi_{T_1}(u) = \mathrm{E}\left[\mathrm{e}^{\mathrm{i}uT_1}\right] = \int_0^\infty \lambda \mathrm{e}^{-(\lambda - \mathrm{i}u)t} dt = \frac{\lambda}{\lambda - \mathrm{i}u},$$

$$\varphi_{S_k}(u) = \mathrm{E}\left[\mathrm{e}^{\mathrm{i}u \sum_{j=1}^k T_j}\right] = \mathrm{E}\left[\prod_{j=1}^k \mathrm{e}^{\mathrm{i}uT_j}\right] \overset{\text{iid}}{=} \mathrm{E}\left[\mathrm{e}^{\mathrm{i}uT_1}\right]^k = \left(\frac{\lambda}{\lambda - \mathrm{i}u}\right)^k.$$

(b) First, note that we can rewrite the memoryless condition as

$$\mathrm{P}[X \geq x + y] = \mathrm{P}[X \geq x]\mathrm{P}[X \geq y] \text{ for any } x, y > 0.$$

Let us define for $x > 0$ the function $h(x) = \mathrm{P}[X \geq x]$.
$\boxed{\Leftarrow}$ We know that if $X \sim \text{Exp}(\lambda)$, we have $h(x) = e^{-\lambda x}$. Then

$$h(x)h(y) = e^{-\lambda x} e^{-\lambda y} = e^{-\lambda(x+y)} = h(x+y).$$

$\boxed{\Rightarrow}$ Suppose that X is a positive and continuous random variable. Then, h is a continuous decreasing function which satisfies $h(x+y) = h(x)h(y)$ for all $x, y > 0$. Since X is positive we know that $\lim_{x \to 0^+} h(x) = 1$. Also, since X is not ∞ a.s., we have $\lim_{x \to \infty} h(x) = 0$. Thus, let us consider $x_0 > 0$ such that $1 > h(x_0) > 0$. It is easy to check that for any n natural number $h(nx_0) = h(x_0)^n$ and $h(x_0/n) = h(x_0)^{1/n}$. This implies that $h(px_0/q) = h(x_0)^{p/q}$ for any p, q natural numbers and thus $h(rx_0) = h(x_0)^r$ for every r positive rational number. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and h is continuous, we conclude that $h(yx_0) = h(x_0)^y$ for all y positive real number. Taking $x = yx_0$ and $\lambda = -\log(h(x_0))/x_0 > 0$ we conclude that $h(x) = e^{-\lambda x}$ for every $x > 0$. Therefore $X \sim \text{Exp}(\lambda)$.

Solution 1.2

(a) For $n \in \mathbb{N}$ set $Y_n := \sum_{i=1}^n |X_i|$ and note that the $|X_i|$ are i.i.d. and independent of N. Hence, we have by the monotone convergence theorem and independence of the X_i and N

$$\begin{aligned}\mathrm{E}[|S_N|] &= \mathrm{E}\Big[\sum_{k=0}^{\infty} |S_k| 1_{\{N=k\}}\Big] \leq \mathrm{E}\Big[\sum_{k=0}^{\infty} Y_k 1_{\{N=k\}}\Big] \\ &= \sum_{k=0}^{\infty} \mathrm{E}[Y_k 1_{\{N=k\}}] = \sum_{k=0}^{\infty} \mathrm{E}[Y_k]\mathrm{E}[1_{\{N=k\}}] \\ &= \mathrm{E}[Y_1]\Big(\sum_{k=0}^{\infty} kP[N=k]\Big) = \mathrm{E}[Y_1]\mathrm{E}[N] < \infty.\end{aligned}$$

We have $\mathrm{E}[S_n] = \mu n$ for all $n \in \mathbb{N}$. As N is independent of $(X_i)_{i\in\mathbb{N}}$, it follows for all $n \in \mathbb{N}_0$

$$\mathrm{E}[S_N 1_{\{N=n\}}] = \mathrm{E}[S_n 1_{\{N=n\}}] = \mathrm{E}[S_n]\mathrm{E}[1_{\{N=n\}}] = \mu n \mathrm{E}[1_{\{N=n\}}] = \mathrm{E}[\mu N 1_{\{N=n\}}]\,.$$

Hence, as N takes values in $\mathbb{N}_0$ so that $\sigma(N)$ is generated by the sets $\{N = n\}$, we have $\mathrm{E}[S_N \,|\, N] = \mu N$. This implies

$$\mathrm{E}[S_N] = \mathrm{E}[\mathrm{E}[S_N|N]] = \mathrm{E}[\mu N] = \mu \mathrm{E}[N]\,.$$

(b) We have $\mathrm{E}[S_n^2] = \sigma^2 n + \mu^2 n^2$ for all $n \in \mathbb{N}$. Since $(S_N)^2 \geq 0$, the conditional expectation $\mathrm{E}[(S_N)^2|N]$ is well-defined. As N is independent of $(X_i)_{i\in\mathbb{N}}$, it follows for all $n \in \mathbb{N}$

$$\begin{aligned}\mathrm{E}[S_N^2 1_{\{N=n\}}] &= \mathrm{E}[S_n^2 1_{\{N=n\}}] \\ &= \mathrm{E}[S_n^2]\mathrm{E}[1_{\{N=n\}}] \\ &= (\sigma^2 n + \mu^2 n^2)\mathrm{E}[1_{\{N=n\}}] \\ &= \mathrm{E}[(\sigma^2 N + \mu^2 N^2) 1_{\{N=n\}}]\,.\end{aligned}$$

This implies $\mathrm{E}[S_N^2 \,|\, N] = \sigma^2 N + \mu^2 N^2$. This leads to

$$\begin{aligned}\mathrm{Var}(S_N) &= \mathrm{E}[S_N^2] - \mathrm{E}[S_N]^2 \\ &= \mathrm{E}[\mathrm{E}[S_N^2 \,|\, N]] - \mathrm{E}[S_N]^2 \\ &= \sigma^2 \mathrm{E}[N] + \mu^2 \mathrm{E}[N^2] - \mu^2 \mathrm{E}[N]^2 \\ &= \sigma^2 \mathrm{E}[N] + \mu^2 \mathrm{Var}(N)\,.\end{aligned}$$

Solution 1.3

(a) For all $r \geq 0$:

$$\mathrm{P}[D > r] = \mathrm{P}[N(B_r) = 0] = \exp(-\lambda \pi r^2)\,.$$

Hence, the distribution function F and the density function f of D are given by

$$F(r) = (1 - \exp(-\lambda\pi r^2))1_{\{r\geq 0\}}\,, \quad f(r) = 2\lambda\pi r \exp(-\lambda\pi r^2)1_{\{r\geq 0\}}.$$

(b) For all $u > r$:

$$\begin{aligned}
\mathrm{P}[N(B_u) = 1 | N(B_r) = 1] &= \frac{\mathrm{P}[N(B_u \setminus B_r) = 0, N(B_r) = 1]}{P[N(B_r) = 1]} \\
&= \frac{\mathrm{P}[N(B_u \setminus B_r) = 0]\mathrm{P}[N(B_r) = 1]}{\mathrm{P}[N(B_r) = 1]} \\
&= \exp(-\lambda\pi(u^2 - r^2))
\end{aligned}$$

Hence, as $r \to 0$ we obtain

$$\mathrm{P}[N(B_u) = 1 | N(B_r) = 1] \to \exp(-\lambda\pi u^2)\,.$$