ETH Zürich, FS 2019
D-MATH
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Applied Stochastic Processes

Exercise sheet 10

Exercise 10.0 Frog Markov chain
Let $(X_n)_{n\geq 0}$ be the Markov chain with state space $\{1,2\}$, initial distribution $\mu = (\mu_1, \mu_2)$ and transition matrix

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}, \text{ where } 0 < p, q \leq 1$$

(a) Compute $\mathrm{P}_\mu[X_n = i]$ for every n.

(b) Deduce the value of $\lim_{n\to\infty} \mathrm{P}_\mu[X_n = i]$.

Exercise 10.1 Let E be a a countable state space and fix $x \in E$. We consider a Markov chain $(X_n)_{n\geq 0}$ under P_x. Assume that $\mathbf{P}_x[H_x^+ < \infty] = 1$. Define $H_x^{(1)} = H_x^+$ and

$$H_x^{(n+1)} = \min\{k > 0;\ X_{T_n+k} = x\} \text{ for } n \geq 1,$$

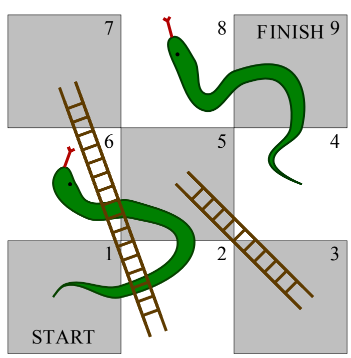
where $T_i = H_x^{(1)} + \cdots + H_x^{(i)}$.

(a) Show that under P_x, the random variables $(H_x^{(i)})_{i\geq 1}$ are i.i.d.

(b) Show that the process defined by $N_t = \sum_{1\leq i\leq t} 1_{\{X_i = x\}}$ is a renewal process.

Exercise 10.2 Let us consider the reflected random walk, that is, the Markov chain with state space $\mathbb{N}_0$ and transition probability given by $p_{0,1} = 1$ and $p_{x,x+1} = \alpha$, $p_{x,x-1} = 1-\alpha$ for $x \geq 1$. Show that for $\alpha \leq 1/2$ all the states are recurrent, and for $\alpha > 1/2$ all the states are transient.

Exercise 10.3 Snakes and ladders.
A simple game of 'snakes and ladders' is played on a board of nine squares.



At each turn a player tosses a fair coin and advances one or two places according to whether the coin lands heads or tails. If you land at the foot of a ladder you climb to the top, but if you land at the head of a snake you slide down to the tail.

(a) How many turns on average does it take to complete the game?
Hint: *Call $k_i = \mathrm{E}_i[H_9]$ and find some relations between the k_i for $i \in \{1, \dots, 9\}$.*

(b) What is the probability that a player who has reached the middle square will complete the game without slipping back to square 1?

Solution 10.0

(a) We know that $\mathbf{P}_\mu[X_n = i] = (\mu P^n)_i$. The eigenvalues of P are 1 and $1-p-q$, and since they are different P is diagonalizable. We can explicitely find the diagonalized form

$$P = \begin{pmatrix} 1 & \frac{-p}{q} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1-p-q \end{pmatrix} \begin{pmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ \frac{-q}{p+q} & \frac{q}{p+q} \end{pmatrix}.$$

Then,

$$P^n = \frac{1}{p+q} \begin{pmatrix} q + p(1-p-q)^n & p - p(1-p-q)^n \\ q - q(1-p-q)^n & p + q(1-p-q)^n \end{pmatrix}.$$

Therefore

$$\mathbf{P}_\mu[X_n = 1] = \frac{1}{p+q} \left(\mu_1(q + p(1-p-q)^n) + \mu_2(q - q(1-p-q)^n) \right),$$
$$\mathbf{P}_\mu[X_n = 2] = \frac{1}{p+q} \left(\mu_1(p - p(1-p-q)^n) + \mu_2(p + q(1-p-q)^n) \right).$$

(b) First, note that $\mu_2 = 1 - \mu_1$. If $p = q = 1$ we have that

$$\mathbf{P}_\mu[X_n = 1] = \frac{1}{2} \left(1 - (-1)^n + 2\mu_1(-1)^n \right),$$
$$\mathbf{P}_\mu[X_n = 2] = \frac{1}{2} \left(1 + (-1)^n - 2\mu_1(-1)^n \right),$$

which does not converges as $n \to \infty$. On the other hand, if $\min\{p,q\} < 1$, we have that $(1-p-q)^n \xrightarrow[n\to\infty]{} 0$. Therefore

$$\lim_{n\to\infty} \mathbf{P}_\mu[X_n = 1] = \frac{q}{p+q}, \quad \lim_{n\to\infty} \mathbf{P}_\mu[X_n = 2] = \frac{p}{p+q},$$

which does not depend on the initial distribution.

Solution 10.1

(a) First, notice that for every $i \geq 1$, T_i is a stopping time. Indeed,

$$\{T_i = n\} = \bigcup_{\substack{I \subset \{1,\dots,n-1\} \\ |I| = i-1}} \{X_j = x, j \in I \cup \{n\}\} \cap \{X_j \neq x, j \in \{1,\dots,n-1\} \backslash I\}.$$

To prove that $(H_x^{(i)})_{i\geq 1}$ are i.i.d. we proceed by induction on i. Suppose that $H_x^{(1)}, \dots, H_x^{(i)}$ are i.i.d. finite almost surely. Let $f_1, \dots, f_{i+1} : \mathbb{N} \cup \{\infty\} \to \mathbb{R}$ measurable bounded functions. Since $f_1(H_x^{(1)}), \dots, f_i(H_x^{(i)}) \in \mathcal{F}_{T_i}$, we have

$$\mathbf{E}_x[f_1(H_x^{(1)}) \cdots f_i(H_x^{(i)}) \cdot f_{i+1}(H_x^{(i+1)})] = \mathbf{E}_x[f_1(H_x^{(1)}) \cdots f_i(H_x^{(i)}) \cdot \mathbf{E}_x[f_{i+1}(H_x^{(i+1)}) | \mathcal{F}_{T_i}]]. \quad (1)$$

We know that $T_i < \infty$ a.s. and that $X_{T_i} = x$ a.s. Also, notice that $f_{i+1}(H_x^{(i+1)}) = f_{i+1}(\min\{k > 0;\ X_{T_i+k} = x\}) = g((X_{T_i+n})_{n\geq 0})$ for some measurable function g. By the strong Markov property, we have that $\mathbf{P}_x$-a.s.

$$\mathbf{E}_x[f_{i+1}(H_x^{(i+1)}) | \mathcal{F}_{T_i}] = \mathbf{E}_x[f_{i+1}(H_x^{(1)})].$$

This shows that $H_x^{(i)}$ has the same distribution as $H_x^{(1)}$. To conclude independence, we use last expression in (1) and the induction hypothesis to conclude that

$$\mathbf{E}_x[f_1(H_x^{(1)}) \cdots f_{i+1}(H_x^{(i+1)})] = \mathbf{E}_x[f_1(H_x^{(1)})] \cdots \mathbf{E}_x[f_{i+1}(H_x^{(1)})].$$

Therefore $H_x^{(1)}, \dots, H_x^{(i+1)}$ are i.i.d. and in particular finite almost surely.

(b) Notice that $N_t = \sum_{1\leq i\leq t} 1_{\{X_i=x\}} = \sum_{i\geq 1} 1_{\{T_i\leq t\}}$, where T_i is the sum of i i.i.d. random variables with $\mathbf{P}_x[H_x^{(1)} = 0] = 0$ by definition. Hence $(N_t)_{t\geq 0}$ is a renewal process.

Solution 10.2

(a) We will compare the reflected random walk with the following biased random walk $(\widetilde{X}_n)_{n\geq 0}$ on $\mathbb{Z}$. Let $(Y_n)_{n\geq 1}$ be a sequence of i.i.d. random variables independent of $(X_n)_{n\in\mathbb{N}}$ under $\mathbf{P}_1$ with

$$\mathbf{P}_0[Y_i = 1] = \alpha = 1 - \mathbf{P}_0[Y_i = -1]. \tag{2}$$

Set $\widetilde{X}_0 = 1$ and $\widetilde{X}_n = 1 + \sum_{i=1}^n Y_i$ for $n \in \mathbb{N}$. Define $\widetilde{H}_0 := \inf\{n \geq 1; \widetilde{X}_n = 0\}$. Noting that $X_1 = 1$ $\mathbf{P}_0$-a.s. we have

$$\mathbf{P}_0[H_0^+ < \infty] = \mathbf{P}_1[H_0 < \infty]. \tag{3}$$

Notice that, $\mathbf{P}_1$-a.s. the process X_n and $\widetilde{X}_n$ have the same distribution before they hit 0. This implies that

$$\mathbf{P}_1[H_0 < \infty] = \mathbf{P}_1[\widetilde{H}_0 < \infty]. \tag{4}$$

Since $(\widetilde{X}_n)_{n\geq 0}$ is a Markov chain on $\mathbb{Z}$ starting at 1, we know by the simple Markov property that

$$\widetilde{\rho}_{1,0} = \mathbf{P}_1[\widetilde{H}_0 < \infty] = (1-\alpha) + \alpha\mathbf{P}_2[\widetilde{H}_0 < \infty].$$

Observe that if we start from 2 we need to hit 1 before hitting 0. Then,

$$\begin{aligned}\mathbf{P}_2[\widetilde{H}_0 < \infty] &= \mathbf{P}_2[\widetilde{H}_0 < \infty, \widetilde{H}_1 < \infty] \\ &= \mathbf{E}_2[\mathbf{E}_2[1_{\{\widetilde{H}_0<\infty\}} 1_{\{\widetilde{H}_1<\infty\}} | \mathcal{F}_{\widetilde{H}_1}]]\end{aligned}$$

By the strong Markov property, we know that

$$\mathbf{E}_2[1_{\{\widetilde{H}_0<\infty\}} 1_{\{\widetilde{H}_1<\infty\}} | \mathcal{F}_{\widetilde{H}_1}] = \mathbf{E}_2[\mathbf{E}_1[1_{\{\widetilde{H}_0<\infty\}}] 1_{\{\widetilde{H}_1<\infty\}}] = \mathbf{E}_2[1_{\{\widetilde{H}_1<\infty\}}]\mathbf{E}_1[1_{\{\widetilde{H}_1<\infty\}}]$$

We also know that the process is stationary, then $\mathbf{E}_2[1_{\{\widetilde{H}_1<\infty\}}] = \mathbf{E}_1[1_{\{\widetilde{H}_0<\infty\}}] = \rho_{1,0}$. Putting all together, we get

$$\widetilde{\rho}_{1,0} = (1-\alpha) + \alpha(\widetilde{\rho}_{1,0})^2.$$

This equation has solutions 1 and $(1-\alpha)/\alpha$. If $\alpha \leq 1/2$ we have $(1-\alpha)/\alpha \geq 1$. Therefore $\mathbf{P}_0[H_0^+ < \infty] = \widetilde{\rho}_{1,0} = 1$ and the state 0 is recurrent for the reflected random walk. If $\alpha < 1/2$ we get that $(1-\alpha)/\alpha < 1$. If $\widetilde{\rho}_{1,0} < 1$ then 0 is a transient state for the reflected random walk. We just need to rule out the possibility that $\widetilde{\rho}_{1,0} = 1$. By the strong law of large numbers, we know that $\mathbf{P}_1$-a.s.

$$\frac{\widetilde{X}_n}{n} = \frac{1 + \sum_{i=1}^n Y_i}{n} \xrightarrow[n\to\infty]{} \mathbf{E}_1[Y_1] = 2\alpha - 1 > 0.$$

In particular $\lim_{n\to\infty} \widetilde{X}_n = +\infty$ $\mathbf{P}_1$-a.s. If $\widetilde{\rho}_{1,0} = 1$, by stationarity we have that $\widetilde{\rho}_{n,n-1} = 1$ for all $n \in \mathbb{Z}$. By the strong Markov property, this means there exists a subsequence of $(\widetilde{X}_n)_{n\geq 0}$ which is arbitrarily negative, which contradicts $\lim_{n\to\infty} \widetilde{X}_n = +\infty$ $\mathbf{P}_1$-a.s.

Solution 10.3

(a) Let us denote by $(X_n)_{n\geq 0}$ the Markov chain with transition probability corresponding to the rules of the game. Recall that $H_i = \inf\{n \geq 0; X_n = i\}$. Let us call $k_i = \mathbf{E}_i[H_9]$ for $i \in \{1, \dots, 9\}$. We observe that 9 is an absorbing state and that $k_9 = 0$. Then we can express H_9 as

$$H_9 = f((X_n)_{n\geq 0}) = \sum_{n=0}^{\infty} 1_{\{X_n<9\}}$$

where f is a measurable function. Then, for $i \in \{1, \dots, 8\}$ we have $\mathbf{P}_i$-a.s. that

$$\begin{aligned} k_i &= \sum_{j=1}^{9} \mathbf{E}_i[H_9 | X_1 = j] \mathbf{P}_i[X_1 = j] \\ &= \sum_{j=1}^{9} \mathbf{E}_i[1_{\{X_0 < 9\}} + f((X_{n+1})_{n \geq 0}) | X_1 = j] p_{i,j} \\ &\overset{(1)}{=} \sum_{j=1}^{9} (1 + \mathbf{E}_j[f((X_n)_{n \geq 0})])\, p_{i,j} \\ &= \sum_{j=1}^{9} (1 + k_j) p_{i,j} \end{aligned}$$

where the equality (1) is justified by the Markov property. Applying this to the model, and considering the effect of the ladders and snakes we get to the following system of equations

$$\begin{aligned} k_1 &= \frac{1}{2}(1 + k_7) + \frac{1}{2}(1 + k_5) \\ k_4 &= \frac{1}{2}(1 + k_5) + \frac{1}{2}(1 + k_1) \\ k_5 &= \frac{1}{2}(1 + k_1) + \frac{1}{2}(1 + k_7) \\ k_7 &= \frac{1}{2}(1 + k_4) + \frac{1}{2}(1 + k_9) \end{aligned}$$

Since $k_9 = 0$ we can solve this system. We obtain that the average number of turns it takes to complete the game is given by $k_1 = 7$.

(b) Notice that the probability that a player starting from the middle square will complete the game without slipping to the square 1 is exactly $\mathbf{P}_5[H_9 < H_1]$. Using the Markov property repeatedly we get

$$\begin{aligned} \mathbf{P}_5[H_9 < H_1] &= p_{5,6} \underbrace{\mathbf{P}_1[H_9 < H_1]}_{=0} + p_{5,7} \mathbf{P}_7[H_9 < H_1] \\ &= \frac{1}{2}(p_{7,8} \mathbf{P}_4[H_9 < H_1] + p_{7,9} \underbrace{\mathbf{P}_9[H_9 < H_1]}_{=1}) \\ &= \frac{1}{2} \left(\frac{1}{2}(p_{4,5} \mathbf{P}_5[H_9 < H_1] + p_{4,6} \underbrace{\mathbf{P}_1[H_9 < H_1]}_{=0}) + \frac{1}{2} \right) \\ &= \frac{1}{8} \mathbf{P}_5[H_9 < H_1] + \frac{1}{4} \end{aligned}$$

Then $\mathbf{P}_5[H_9 < H_1] = 2/7$.