Applied Stochastic Processes

Exercise sheet 11

Exercise 11.1 Recurrence and stationarity.
Consider the Markov chain $(X_n)_{n\in\mathbb{N}}$ with state space $\mathbb{N} := \{1, 2, ...\}$, and transition probability

$$p_{i,j} = \begin{cases} \pi(j) & \text{if } i = 1,\ j \geq 1, \\ 1 & \text{if } i > 1,\ j = i-1, \\ 0 & \text{else,} \end{cases}$$

where π denotes a probability distribution on $\mathbb{N}$ with $\sum_{i\in\mathbb{N}} i\pi(i) < \infty$.

(a) Determine the positive recurrent, null recurrent, and transient states.

(b) Find a stationary distribution ν for this Markov chain. Is this the unique stationary distribution?

Exercise 11.2 Let $(X_n)_{n\geq 0}$ a Markov chain on a countable state space E. Let $x, y \in E$ such that $x \longleftrightarrow y$. Prove that x is positive recurrent if and only if y is positive recurrent.

Exercise 11.3 Simple Random Walk on $\mathbb{Z}^d$ and Fourier Analysis.
Let $(X_n)_{n\geq 0}$ be the simple random walk on $\mathbb{Z}^d$, $d \geq 1$. Suppose that $X_0 = 0$ and let V_0 be the total number of returns to 0. For $\xi \in [-\pi, \pi]^d$, we denote the characteristic function X_1 by $\varphi(\xi) = \mathbf{E}_0[\exp(i\xi \cdot X_1)]$.

(a) Show that

$$\mathbf{P}_0[X_n = 0] = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \varphi(\xi)^n d\xi.$$

(b) Show that

$$\mathbf{E}_0[V_0] = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{\varphi(\xi)}{1-\varphi(\xi)} d\xi.$$

(c) Calculate explicitly $\varphi(\xi)$ and show that

$$\frac{1}{4d}|\xi|^2 \leq 1 - \varphi(\xi) \leq \frac{1}{2d}|\xi|^2.$$

(d) Conclude that 0 is a recurrent state if and only if $d \in \{1, 2\}$.

Solution 11.1

(a) First, note that the state $1 \in \mathbb{N}$ is recurrent. Indeed,

$$\rho_{1,1} := \mathbf{P}_1\left[\bigcup_{n=1}^{\infty}\{X_n = 1\}\right] = \sum_{i\in\mathbb{N}} \pi(i) = 1. \tag{1}$$

Moreover, if the Markov chain jumps to state i starting from 1, then it will return to state 1 in exactly i steps. Hence $\mathbf{E}_1[H_1] = \sum_{i\in\mathbb{N}} i\pi(i) < \infty$ by assumption. This implies that the state 1 is positive recurrent.

On the other hand, let us define $m := \sup\{i \in \mathbb{N} : \pi(i) > 0\}$. We have two cases:

- If $m = \infty$, then all states $i \in \mathbb{N}$ are in the communicating class of 1 and thus they are positive recurrent (see Exercise 11.2).
- If $m < \infty$, then all states $i \in \{1, 2, \dots, m\}$ are in the communicating class of 1 and thus positive recurrent. The states $i \in \mathbb{N} \setminus \{1, 2, \dots, m\}$ are transient, since for all $i \in\in \mathbb{N} \setminus \{1, 2, \dots, m\}$ (defining $\rho_{i,j}$ similarly to (1)),

$$\rho_{i,i} = p_{i,i-1} \cdot \ldots \cdot p_{2,1}\rho_{1,i} = \rho_{1,i} = 0 < 1.$$

(b) Let X be an integer-valued random variable with distribution π. By assumption,

$$E[X] = \sum_{i\in\mathbb{N}} i\pi(i) < \infty.$$

We define a distribution $(\nu_i)_{i\in\mathbb{N}}$ by

$$\nu_i := \frac{P[X \geq i]}{E[X]}, \quad i \in \mathbb{N}.$$

To show that $(\nu_i)_{i\in\mathbb{N}}$ is a stationary distribution, we observe that

$$\begin{aligned}
\sum_{i\in\mathbb{N}} \nu_i p_{i,j} &= \nu_1 p_{1,j} + \nu_{j+1} p_{j+1,j} \\
&= \frac{P[X \geq 1]}{E[X]}\pi(j) + \frac{P[X \geq j+1]}{E[X]} \\
&= (1 \cdot P[X = j] + P[X \geq j+1])/E[X] \\
&= P[X \geq j]/E[X] \\
&= \nu_j.
\end{aligned}$$

Since the Markov chain is irreducible, conclude that ν is its unique stationary probability distribution.

Solution 11.2 Assume that x is positive recurrent. Since

$$\frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i = x\}} \xrightarrow[n\to\infty]{} \frac{1}{\mathbf{E}_x[H_x^+]}, \quad \mathbf{P}_x\text{-a.s.}$$

we have by the dominated convergence theorem, that

$$\frac{1}{n}\sum_{i=1}^{n} p_{x,x}^{(i)} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{P}_x[X_i = x] \xrightarrow[n\to\infty]{} \frac{1}{\mathbf{E}_x[H_x^+]}.$$

Since $x \longleftrightarrow y$, we can find k, ℓ such that $p_{y,x}^{(k)}, p_{x,y}^{(\ell)} > 0$. By the Chapman-Kolmogorov equation, for all integer $j \geq 0$ we have that $p_{y,y}^{(k+j+\ell)} \geq p_{y,x}^{(k)} \cdot p_{x,x}^{(j)} \cdot p_{x,y}^{(\ell)}$. Hence

$$\underbrace{\frac{1}{n}\sum_{i=1}^{n} p_{y,y}^{(i)}}_{\xrightarrow[n\to\infty]{} \frac{1}{\mathbf{E}_y[H_y^+]}} \geq \frac{1}{n}\sum_{j=1}^{n-k-\ell} p_{y,y}^{(k+j+\ell)} \geq p_{y,x}^{(k)} \underbrace{\left(\frac{1}{n}\sum_{j=1}^{n-k-\ell} p_{x,x}^{(j)}\right)}_{\xrightarrow[n\to\infty]{} \frac{1}{\mathbf{E}_x[H_x^+]}} p_{x,y}^{(\ell)}.$$

Therefore, $\mathbf{E}_y[H_y^+] < \infty$ and y is positive recurrent.

Solution 11.3

(a) Let us denote by φ_n the characteristic function of X_n. Since X_n is the sum of n i.i.d. random variables with distribution X_1, we have that $\varphi_n(\xi) = \varphi(\xi)^n$. Then, by the Fourier inversion formula we obtain that

$$\mathbf{P}_0[X_n = 0] = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \exp(i\xi \cdot 0)\varphi_n(\xi)d\xi = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \varphi(\xi)^n d\xi.$$

(b) Notice that

$$\begin{aligned}
\mathbf{E}_0[V_0] &= \sum_{n\geq 1} \mathbf{P}_0[X_n = 0] \\
&= \sum_{n\geq 1} \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \varphi(\xi)^n d\xi \\
&= \frac{1}{(2\pi)^d} \lim_{\alpha\to 1^-} \sum_{n\geq 1} \alpha^n \int_{[-\pi,\pi]^d} \varphi(\xi)^n d\xi \\
&= \frac{1}{(2\pi)^d} \lim_{\alpha\to 1^-} \sum_{n\geq 1} \int_{[-\pi,\pi]^d} (\alpha\varphi(\xi))^n d\xi \\
&= \frac{1}{(2\pi)^d} \lim_{\alpha\to 1^-} \int_{[-\pi,\pi]^d} \left(\sum_{n\geq 1} (\alpha\varphi(\xi))^n\right) d\xi
\end{aligned}$$

where the exchange of the series and the integral is justified by the dominated convergence theorem (domination by the convergent series $\sum_n \alpha^n, \alpha < 1$). Then

$$\mathbf{E}_0[V_0] = \frac{1}{(2\pi)^d} \lim_{\alpha\to 1^-} \int_{[-\pi,\pi]^d} \frac{\alpha\varphi(\xi)}{1-\alpha\varphi(\xi)} = \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} \frac{\varphi(\xi)}{1-\varphi(\xi)}.$$

This time the exchange of the limit and the integral is justified by the monotone convergence theorem, applied separately over the sets $\{\varphi < 0\}$ and $\{\varphi \geq 0\}$.

(c) Let us denote $\xi = (\xi_1, \dots, \xi_d)$. If $e_1, \dots, e_d$ is the canonical base of $\mathbb{Z}^d$ and we denote $e_{-k} = -e_k$, we see that

$$\varphi(\xi) = \sum_{k\in\{-d,\dots,d\}} \exp(i\xi \cdot e_k)\mathbf{P}_0[X_1 = e_k] = \frac{1}{2d}\sum_{j=1}^{d}(e^{i\xi_j} + e^{-i\xi_j}) = \frac{1}{d}\sum_{i=1}^{d}\cos(\xi_j).$$

Using the Taylor expansion of $\cos(x) = 1 - x^2/2 + O(x^4)$ we can prove that $x^2/2 \leq 1 - \cos(x) \leq x^2$. Applying this to $x = \xi_j$, summing over j and using $|\xi|^2 = \xi_1^2 + \dots + \xi_d^2$, we obtain

$$\frac{1}{4d}|\xi|^2 \leq 1 - \varphi(\xi) \leq \frac{1}{2d}|\xi|^2.$$

(d) Now, notice that the transience or recurrence of 0 depends on the convergence or not of the integral $\int_{[-\pi,\pi]^d} \frac{\varphi(\xi)}{1-\varphi(\xi)} d\xi$. Using the previous inequality this question translates in the convergence or not of the integral $\int_{[-\pi,\pi]^d} \frac{d\xi}{|\xi|^2}$, which we know is equal to $+\infty$ if $d \in \{1,2\}$ and that is $< +\infty$ if $d \geq 3$. This concludes the proof.