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Applied Stochastic Processes

Exercise sheet 12

Exercise 12.1

(a) Find all the stationary probability measures of the reflected random walk on $\mathbb{Z}$.

(b) Consider the biased random walk on the circle $\mathbb{Z}/n\mathbb{Z}$, i.e. the Markov chain with transition probability given by $p_{x,x+1} = \alpha$ and $p_{x,x-1} = 1-\alpha$ for all $x \in \mathbb{Z}/n\mathbb{Z}$. Show that there exists a reversible probability measure for this Markov chain if and only if $\alpha = 1/2$.

Exercise 12.2 Let $(X_n)_{n\geq 0}$ a Markov chain on a countable state space E. Let us denote $R_1^+, R_2^+, \ldots$ its positive recurrent classes.

(a) For $i = 1, 2, \ldots$ we consider the process $(X_n^{(i)})_{n\geq 0}$ with state space R_i^+ and transition probability given by $(p_{x,y})_{x,y\in R_i^+}$, where p is the transition probability for $(X_n)_{n\geq 0}$. Show that $(X_n^{(i)})_{n\geq 0}$ is a Markov chain on R_i^+. Show that there is a unique stationaty probability measure π_i for the chain $(X_n^{(i)})_{n\geq 0}$.

(b) Show that if π is a stationary probability measure for $(X_n)_{n\geq 0}$, then there exists $\alpha_1, \alpha_2, \ldots \in [0,1]$ with $\sum_{i\in\mathbb{N}} \alpha_i = 1$, such that $\pi = \sum_{i\in\mathbb{N}} \alpha_i \pi_i$.

Exercise 12.3 We consider the following modification of the *Ehrenfest* model of diffusion: N molecules are distributed in containers (I) and (II). At each step, we choose at random one molecule and place it

- into (I) with a probability $\frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}$,
- into (II) with a probability $\frac{1}{1+e^{-\beta\varepsilon}}$,

where $\beta > 0$ is the inverse temperature and $\varepsilon > 0$ an energy penalty.

(a) Let X_n denote the number of molecules in (I) at step n (starting from some initial distribution $X_0 \sim \mu$). Show that $(X_n)_{n\geq 0}$ is a Markov chain and find its transition probability.

(b) Show that the binomial distribution with parameters N and $\frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}$ is a reversible probability measure for $(X_n)_{n\geq 0}$.

(c) How many molecules will be on average in (I) as $n \to \infty$? What is the high temperature and low temperature behaviour of this quantity?

Solution 12.1

(a) Let us suppose that π is a stationary distribution of the reflected random walk on $\mathbb{Z}$. If we denote the transition probability of this chain by $(p_{x,y})_{x,y\in\mathbb{Z}}$, we have that for every $x \in \mathbb{Z}$

$$\sum_{y\in\mathbb{Z}} \pi(y) p_{y,x} = \pi(x).$$

Using that $p_{0,1} = 1$, $p_{x,x+1} = \alpha$, $p_{x,x-1} = 1-\alpha$ for $x \geq 1$ and $p_{x,y} = 0$ otherwise, we obtain the following system of equations

$$\begin{aligned}(1-\alpha)\pi(1) &= \pi(0)\\ (1-\alpha)\pi(2) + \pi(0) &= \pi(1)\\ (1-\alpha)\pi(x+1) + \alpha\pi(x-1) &= \pi(x), \text{ for } x \geq 2.\end{aligned}$$

This implies that $\pi(x) = \pi(0)\frac{\alpha^{x-1}}{(1-\alpha)^x}$. Since π is a probability distribution we have that

$$1 = \pi(0)\alpha^{-1} \sum_{x\geq 0} \left(\frac{\alpha}{1-\alpha}\right)^x. \tag{1}$$

We know that this series converges if and only if $\alpha < 1/2$. If $\alpha \geq 1/2$ then there is no stationary distribution (note we already knew that since in this case the reflected random walk is either transient or null recurrent for any starting point). In the positive recurrent case ($\alpha < 1/2$) equation (1) gives us $\pi(0) = \frac{\alpha(1-2\alpha)}{1-\alpha}$, and therefore $\pi(x) = \frac{\alpha^x(1-2\alpha)}{(1-\alpha)^{x+1}}$ for $x \geq 1$.

(b) As before, suppose that π is a stationary probability measure for this chain. Then

$$\pi(x) = \alpha\pi(x-1) + (1-\alpha)\pi(x+1) \text{ for all } x \in \mathbb{Z}/n\mathbb{Z}.$$

This implies that $\pi(x) - \pi(x-1) = \frac{1-\alpha}{\alpha}(\pi(x+1) - \pi(x))$ for all $x \in \mathbb{Z}/n\mathbb{Z}$. If $\alpha \neq 1/2$, then the sequence of increments $\pi(x) - \pi(x-1)$ is strict monotone, which is a contradiction since our state space is the 1 dimensional torus. Therefore $\alpha = 1/2$. Reciprocally, if $\alpha = 1/2$ it is direct to show that the uniform distribution is a stationary probability measure for this Markov chain.

Solution 12.2

(a) To show that $(X_n^{(i)})_{n\geq 0}$ is a Markov chain on R_i^+ it suffices to show that $(p_{x,y})_{x,y\in R_i^+}$ is in fact a transition probabilty. Since R_i^+ is a positive recurrent class we know that $p_{x,y} = 0$ for $x \in R_i^+$ and $y \in E \setminus R_i^+$. Then, using the fact that p is a transition probability, we obtain that for every $x \in R_i^+$

$$1 = \sum_{y\in E} p_{x,y} = \sum_{y\in R_i^+} p_{x,y},$$

which shows what we wanted. Since $(X_n^{(i)})_{n\geq 0}$ runs in a single communicating class, it is irreducible. This implies it has a unique stationary probability measure π_i.

(b) If π is a stationary probability measure for $(X_n)_{n\geq 0}$, we know that for every $n \geq 1$,

$$\pi(x) = \mathbf{P}_\pi[X_n = x] = \sum_{y\in E} \pi(y)\mathbf{P}_y[X_n = x].$$

If x is a transient state, then $\mathbf{P}_y[X_n = x] = 0$ for every y recurrent state. If y is a transient state, we also have that $\mathbf{E}_y[V_x] = \sum_{n\geq 0} \mathbf{P}_y[X_n = x] < \infty$. This implies that $\mathbf{P}_y[X_n = x] \to 0$

as $n \to \infty$. We conclude that $\pi(x) = 0$ for every x transient state. If the chain is started from a recurrent class, we know it will stay forever in that class. As in part (a), we can prove that the chain $(X_n)_{n\geq 0}$ restricted to a null recurrent class is still a Markov chain. This Markov chain will remain null recurrent, and therefore $\pi(x) = 0$ for every x null recurrent state. We conclude then that π concentrates its mass on the positive recurrent classes. Moreover, if $x \in R_i^+$, we have that $\pi(x) = \sum_{y\in R_i^+} \pi(x)\mathbf{P}_y[X_n = x]$, which means that $\pi(x) = \alpha_i\pi_i(x)$, by uniqueness of the stationary measure π_i. Since π is a probability distribution, we conclude that $\pi = \sum_i \alpha_i\pi_i$ with $\alpha \in [0, 1]$ and $\sum_i \alpha_i = 1$.

Solution 12.3

(a) Assume that at step $n \geq 0$, there are $X_n \in \{0, ..., N\}$ molecules in container (I).

- A molecule from (I) is selected with probability $\frac{X_n}{N}$ and put into (I) with probability $\frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}$, and put into (II) with probability $\frac{1}{1+e^{-\beta\varepsilon}}$
- Analogously, a molecule from (I) is selected with probability $\frac{N-X_n}{N}$ and put into (I) with probability $\frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}$, and put into (II) with probability $\frac{1}{1+e^{-\beta\varepsilon}}$

This leads to the following transition probability:

$$p_{x,y} = \begin{cases} \frac{x}{N}\frac{1}{1+e^{-\beta\varepsilon}}, & y = x-1, \\ \frac{N-x}{N}\frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}, & y = x+1, \\ \frac{x}{N}\frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}} + \frac{N-x}{N}\frac{1}{1+e^{-\beta\varepsilon}}, & y = x, \\ 0, & y \notin \{x-1, x, x+1\} \end{cases}$$

(b) Let π be the binomial distribution with parameters N and $\frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}$. The case $y = x$ in the detailed balance condition is trivial, so we consider $y = x + 1$:

$$\begin{aligned} \pi(x)p_{x,x+1} &= \binom{N}{x}\left(\frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}\right)^x\left(\frac{1}{1+e^{-\beta\varepsilon}}\right)^{N-x}\frac{N-x}{N}\frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}} \\ &= \frac{(N-1)!}{x!(N-x-1)!}\frac{e^{-\beta(x+1)\varepsilon}}{(1+e^{-\beta\varepsilon})^{N+1}} \\ &= \binom{N}{x+1}\left(\frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}\right)^{x+1}\left(\frac{1}{1+e^{-\beta\varepsilon}}\right)^{N-x-1}\frac{x+1}{N}\frac{1}{1+e^{-\beta\varepsilon}} = \pi(x+1)p_{x+1,x}. \end{aligned}$$

By replacing $x \rightsquigarrow x - 1$, this also shows the detailed balance condition in the case $y = x - 1$. Thus, π is reversible for $(p_{x,y})_{x,y\in\{0,...,N\}}$.

(c) The average number of molecules in container (I) up to and including time n is given by

$$Z_n = \frac{1}{n+1}\sum_{i=0}^{n} X_n$$

We remark that the Markov chain $(X_n)_{n\geq 0}$ is irreducible and thus π is its unique stationary distribution. We can thus apply the ergodic theorem for Markov chains for $\iota : \{0, ..., N\} \to \mathbb{R}$ the canonical inclusion, which is bounded (and measurable) and conclude:

$$Z_n = \frac{X_0 + ... + X_n}{n+1} = \frac{\iota(X_0) + ... + \iota(X_n)}{n+1} \to \int \iota d\pi = \sum_{k=0}^{N} k\pi(k) = E[X], \mathbf{P}_\mu\text{-a.s.},$$

where $X \sim \pi$, so $E[X] = \frac{Ne^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}$.

- In the zero temperature limit, $\beta \to \infty$, and $E[X] \to 0$, so the container (I) will be empty on average.
- In the high temperature limit, $\beta \simeq 0$, and $E[X] = \frac{N}{1+e^{\beta\varepsilon}} \simeq \frac{N}{2+\beta\varepsilon+O((\beta\varepsilon)^2)}$, which is close to $N/2$ (and in fact equal to $N/2$ for $\beta = 0$). Thus, in the high temperature limit, both containers will contain on average the same number of molecules.