Applied Stochastic Processes

Exercise sheet 13

In the following exercises we consider $(X_n)_{n\geq 0}$ a Markov chain with transition probability p on a countable state space E.

Exercise 13.1 Assume that $(X_n)_{n\geq 0}$ is irreducible and aperiodic. Prove that for all $x, y \in E$ there exists n_0 such that for all $n \geq n_0$ we have $p^{(n)}_{x,y} > 0$.

Exercise 13.2 Lazy Markov chain
Let us define for all $x, y \in E$,

$$q_{x,y} = \frac{1}{2}\delta_{x,y} + \frac{1}{2}p_{x,y}.$$

(a) Prove that q is a transition probability. We define $(\widetilde{X}_n)_{n\geq 0}$ the Markov chain with this transition probability, which is called the *lazy* version of $(X_n)_{n\geq 0}$.

(b) Assume that $(X_n)_{n\geq 0}$ is irreducible. Prove that $(\widetilde{X}_n)_{n\geq 0}$ is irreducible and aperiodic.

(c) Assume that $(X_n)_{n\geq 0}$ is positive recurrent. Prove that $(\widetilde{X}_n)_{n\geq 0}$ is positive recurrent. What is the stationary distribution of $(\widetilde{X}_n)_{n\geq 0}$?

Exercise 13.3

(a) Assume that $(X_n)_{n\geq 0}$ is irreducible. Show that there is an integer $d \geq 1$ and a partition

$$E = C_0 \cup C_1 \cup \cdots \cup C_{d-1}$$

such that (setting $C_{nd+r} = C_r$)

(i) $p^{(n)}_{x,y} > 0$ only if $x \in C_r$ and $y \in C_{r+n}$ for some $r \in \{0, \dots, d-1\}$;

(ii) $p^{(nd)}_{x,y} > 0$ for all sufficiently large n, for all $x, y \in C_r$, for all $r \in \{0, \dots, d-1\}$.

(b) Let λ be a probability measure on E with $\sum_{x\in C_0} \lambda_x = 1$. Show that for $r \in \{0, 1, \dots, d-1\}$ and $y \in C_r$, one has

$$\mathbf{P}_\lambda[X_{nd+r} = y] \to \frac{d}{\mathbf{E}_y[H_y^+]}, \quad \text{as } n \to \infty.$$

Hint. *Set $Y_n = X_{nd+r}$ and conclude from (a) that the transition probability associated to $(Y_n)_{n\geq 0}$ is irreducible and aperiodic.*

Solution 13.1 Let z be an aperiodic state and $x, y \in E$ arbitrary states. Since the chain is irreducible, there exists $\ell, m > 0$ such that $p_{x,z}^{(\ell)}, p_{z,y}^{(m)} > 0$. Using Chapman-Kolmogorov we can deduce that for all n

$$p_{x,y}^{(\ell+n+m)} \geq p_{x,z}^{(\ell)} p_{z,z}^{(n)} p_{z,y}^{(m)}$$

Then it is enough to show that there exists n_0 such that for all $n \geq n_0$ we have that $p_{z,z}^{(n)} > 0$. To this purpose we will prove the following

Lemma. *Let $A \subset \mathbb{N} \setminus \{0\}$ such that for all $a, b \in A$, $a + b \in A$. Then, $\gcd(A) = 1$ if and only if there exists $n_0 \in \mathbb{N}$ such that $\{n : n \geq n_0\} \subset A$.*

Proof: The reciprocal is trivial, since $\gcd(n_0, n_0 + 1) = 1$. Let us consider an arbitrary element $a \in A$. Let us consider its prime factorization $a = \prod_{i=1}^{n} p_i^{\alpha_i}$, with $n \in \mathbb{N}$. By definition of A, we can find $b_i \in A$ such that p_i does not divide b_i. Doing this for every prime in the factorization of a, we find a finite set $\{a, b_1, \ldots, b_n\}$ with greatest common divisor 1. We will show that starting from this family we can find an element $b \in A$ such that $\gcd(a, b) = 1$. By Bézout's identity there exist $u, v \in \mathbb{Z}$ such that $ub_1 + vb_2 = \gcd(b_1, b_2)$. Choose k such that $kb_3 \geq \max\{-u, -v\}$. Then $b_2' := (kb_3 + u)b_1 + (kb_3 + v)b_2 \in A$ and $\gcd(b_2', b_3) = \gcd(b_1, b_2, b_3)$, which implies that

$$\gcd(a, b_2', b_3, \ldots, b_n) = \gcd(a, b_1, \ldots, b_n) = 1.$$

Repeating this arguments $n - 1$ times we obtain $b \in A$ such that $\gcd(a, b) = 1$. Let us suppose, without loss of generality, that $a < b$. Then the set $B = \{b, 2b, \ldots, ab\} \subset A$ covers all residue classes modulo a, which implies that $B + \{ka : k \in \mathbb{N}\}$ includes every number $\geq ab.$, so we can take $n_0 = ab$ and the lemma is proven.

□

Let us define $A_z = \{n \in \mathbb{N} : p_{z,z}^{(n)} > 0\}$. Since $p_{z,z}^{(n+m)} \geq p_{z,z}^{(n)} p_{z,z}^{(m)}$, we have that A_z satisfies the hypothesis of the lemma, and we are done.

Solution 13.2

(a) Note that for every $x \in E$ we have

$$\sum_{y \in E} q_{x,y} = \frac{1}{2}\delta_{x,x} + \frac{1}{2}\sum_{y \in E} p_{x,y} = 1.$$

Since $q_{x,y} \geq 0$ for every $x, y \in E$ we conclude q is a transition probability.

(b) Let $x, y \in E$ different states. Since p is irreducible we know there exists $x = x_0, x_1, \ldots, x_n = y$ with $p_{x_i, x_{i+1}} > 0$ and such that all the x_i's are different. This implies that

$$q_{x,y}^{(n)} \geq q_{x_0,x_1} \cdots q_{x_{n-1},x_n} = \frac{1}{2^n} p_{x_0,x_1} \cdots p_{x_{n-1},x_n} > 0,$$

i.e., q is irreducible. Since $q_{x,x} \geq \frac{1}{2} > 0$, we conclude that q is aperiodic.

(c) Let us denote $\widetilde{H}_x^+$ the returning time to x. We know that

$$\begin{aligned}
\mathbf{E}_x[\widetilde{H}_x^+] &= \sum_{n \geq 0} \mathbf{P}_x[\widetilde{H}_x^+ > n] \\
&= \sum_{n \geq 1} \sum_{\substack{x = x_0, x_1, \ldots, x_n \in E \\ x_1, \ldots, x_n \neq x}} q_{x_0,x_1} \cdots q_{x_{n-1},x_n}
\end{aligned}$$

$$= \sum_{n\geq 1} \sum_{\substack{x=x_0,x_1,\dots,x_n\in E \\ x_1,\dots,x_n\neq x}} \frac{1}{2^n} p_{x_0,x_1}\cdots p_{x_{n-1},x_n}$$

$$= \sum_{n\geq 1} \frac{1}{2^n}\mathbf{P}_x[H_x^+ > n] \leq \mathbf{E}_x[H_x^+] < \infty$$

since p is positive recurrent. This implies that q is also positive recurrent. From this we can conclude that q has a stationary measure $\widetilde{\pi}$. If we denote π the stationary measure of p, we can deduce from the previous inequality that for all $x \in E$

$$\widetilde{\pi}(x) = \frac{1}{\mathbf{E}_x[\widetilde{H}_x^+]} \geq \frac{1}{\mathbf{E}_x[H_x^+]} = \pi(x).$$

Let us suppose that for some $y \in E$, $\widetilde{\pi}(y) > \pi(y)$. This implies that $\sum_x \widetilde{\pi}(x) > \sum_x \pi(x) = 1$, which is a contradiction, since $\widetilde{\pi}$ is a probability distribution. Therefore $\widetilde{\pi} = \pi$, i.e. the lazy chain has the same stationary distribution as the original chain.

Solution 13.3

(a) Fix a state $z \in E$ and consider $S = \{n \geq 0 : p_{z,z}^{(n)} > 0\}$. Choose $n_1, n_2 \in S$ with $n_1 < n_2$ and such that $d := n_2 - n_1$ is minimal. Define for $r = 0, \dots, d-1$,

$$C_r = \{x \in E : p_{z,x}^{(nd+r)} > 0 \text{ for some } n \geq 0\}.$$

Then $C_0 \cup \cdot \cup C_{d-1} = E$, by irreducibility. Moreover, if $p_{z,x}^{(nd+r)} > 0$ and $p_{z,x}^{(nd+s)} > 0$ for some $r, s \in \{0, 1, \dots, d-1\}$, then, choosing $m \geq 0$ so that $p_{x,z}^{(m)} > 0$, we have $p_{z,z}^{(nd+r+m)} > 0$ and $p_{z,z}^{(nd+s+m)} > 0$ so $r = s$ by minimality of d. Hence we have a partition of E.
To prove (i) suppose that $p_{x,y}^{(n)} > 0$ and $x \in C_r$. Choose m so that $p_{z,x}^{(md+r)} > 0$ then $p_{z,y}^{(md+r+n)} > 0$ so $y \in C_{r+n}$, as required. By taking $x = y = z$ we see that d must divide every element of S, in particular n_1. Now for $nd \geq n_1^2$, we can write $nd = qn_1 + r$ for integers $q \geq n_1$ and $0 \leq r \leq n_1 - 1$. Since d divides n_1 we then have $r = md$ for some integer m and then $nd = (q-m)n_1 + mn_2$. Hence

$$p_{z,z}^{(nd)} \geq \left(p_{z,z}^{(n_1)}\right)^{q-m} \left(p_{z,z}^{(n_2)}\right)^{m} > 0$$

and hence $nd \in S$. To prove (ii) for $x, y \in C_r$ choose m_1 and m_2 so that $p_{x,z}^{(m_1)} > 0$ and $p_{z,y}^{(m_2)} > 0$, then

$$p_{x,y}^{(m_1+nd+m_2)} \geq p_{x,z}^{(m_1)} p_{z,z}^{(nd)} p_{z,y}^{(m_2)} > 0$$

whenever $nd \geq n_1^2$. Since $m_1 + m_2$ is then necessarily a multiple of d, we are done.

(b) Set $\nu(x) = \sum_{y\in C_0} \lambda(y) p_{y,x}^{(r)}$. By part (a) item (i) we have that $\sum_{x\in C_r} \nu(x) = 1$. By definition, we know that Y_n is a $MC(\nu, p^{(d)})$ with state space C_r. Part (a) item (ii) implies that $(Y_n)_{n\geq 1}$ is irreducible and aperiodic. Note that for $y \in C_r$,

$$\mathbf{E}_y[H_y^+] = \mathbf{E}_y\left[\sum_{n\geq 1} 1_{\{X_n\neq y\}}\right] = \mathbf{E}_y\left[d\sum_{n\geq 1} 1_{\{X_{nd+r}=y\}}\right]$$

This means the expected returning time of $(Y_n)_{n\geq 0}$ started from ν is $\mathbf{E}_y[H_y^+]/d$. Since $(Y_n)_{n\geq 0}$ is irreducible and aperiodic, the law of Y_n converges to its unique stationary distribution, which is $\pi(y) = d/\mathbf{E}_y[H_y^+]$. In other words

$$\mathbf{P}_\lambda[X_{nd+r} = y] = \mathbf{P}_\nu[Y_n = y] \to \frac{d}{\mathbf{E}_y[H_y^+]}, \quad \text{as } n \to \infty.$$