Applied Stochastic Processes

Exercise sheet 2

Exercise 2.1 The Waiting Time Paradox.

(a) Let $(N_t)_{t\geq 0}$ be a standard Poisson process with rate $\lambda > 0$. Let $(S_n)_{n\in\mathbb{N}}$ be the arrival times for this process. For a fixed $t > 0$, let $A_t = t - S_{N_t}$ be the time passed after the most recent arrival (or after 0) in the process, and let $B_t = S_{N_t+1} - t$ be the time forward to the next arrival. Let $T_1 \sim \mathrm{Exp}(\lambda)$. Show that A_t and B_t are independent, that B_t is distributed as T_1 and that A_t is distributed as $T_1 \wedge t$.

(b) Let $L_t = A_t + B_t = S_{N_t+1} - S_{N_t}$ be the length of the interarrival interval covering t. Show that L_t has density

$$f_t(x) = \begin{cases} \lambda^2 x e^{-\lambda x} & \text{if } 0 < x < t, \\ \lambda(1+\lambda t)e^{-\lambda x} & \text{if } x \geq t. \end{cases}$$

Show that $\mathrm{E}[L_t]$ converges to $2\mathrm{E}[T_1]$ as $t \to \infty$. Since L_t is the time between two consecutive arrivals, we would expect $\mathrm{E}[L_t] = \mathrm{E}[T_1]$. Give an intuitive resolution of the apparent paradox.

Exercise 2.2 Compound Poisson process.
Let $(N_t)_{t\geq 0}$ be a standard Poisson process with rate $\lambda > 0$ and $(X_k)_{k\in\mathbb{N}}$ a sequence of real-valued i.i.d. random variables with common distribution μ such that $(N_t)_{t\geq 0}$ and $(X_k)_{k\in\mathbb{N}}$ are independent. Define the process $Z = (Z_t)_{t\geq 0}$ by

$$Z_t := \sum_{k=1}^{N_t} X_k, \quad t \geq 0.$$

Z is called a *compound Poisson process* with rate λ and *jump size distribution* μ.

(a) For $t > 0$ determine the characteristic function of Z_t.

(b) Prove that Z has stationary and independent increments.

(c) Show that if $\mathrm{P}[X_i = 1] = 1 - \mathrm{P}[X_i = 0] = p$, then Z is a Poisson process with rate λp.

Exercise 2.3 Let $(N_t)_{t\geq 0}$ be a standard Poisson process with rate $\lambda > 0$. For every $n \in \mathbb{N}$ let $(X_i^{(n)})_{i\in\mathbb{N}_0}$ be a sequence of i.i.d. random variables with distribution Bernoulli(λ/n). Define

$$N_t^{(n)} = \sum_{i=0}^{\lfloor nt \rfloor} X_i^{(n)}, \quad t \geq 0.$$

Show that for all k, for all $0 \leq t_1 < \cdots < t_k < \infty$ and for all $f : \mathbb{N}^k \to \mathbb{R}$ bounded, we have that

$$\mathrm{E}[f(N_{t_1}^{(n)}, \ldots, N_{t_k}^{(n)})] \xrightarrow[n\to\infty]{} \mathrm{E}[f(N_{t_1}, \ldots, N_{t_k})]. \tag{1}$$

Hint: *Prove that for all* $(i_1, \ldots, i_k) \in \mathbb{N}^k$

$$\mathrm{P}[N_{t_1}^{(n)} = i_1, \ldots, N_{t_k}^{(n)} = i_k] \xrightarrow[n\to\infty]{} \mathrm{P}[N_{t_1} = i_1, \ldots, N_{t_k} = i_k]$$

and that this implies (1).

Solution 2.1 Let $t > 0$ be fixed.

(a) Let us consider $0 \leq u \leq t$, $0 \leq v$. We have that

$$P[A_t \geq u, B_t > v] = P[S_{N_t} \leq t - u, S_{N_t+1} > t + v] = P[N_{t+v} - N_{t-u} = 0] = e^{-\lambda u}e^{-\lambda v}$$

and if $u > t$ we have $P[A_t \geq u, B_t > v] = 0$. Let U, V be independent random variables with distribution $\mathrm{Exp}(\lambda)$. Note that

$$P[U \wedge t \geq u, V > v] = e^{-\lambda u}e^{-\lambda v}1_{\{u \leq t\}}.$$

Denote $P^{(A_t,B_t)}$ the joint law of A_t and B_t, and $P^{(U\wedge t,V)} = P^{U\wedge t} \otimes P^V$ the joint law of $U \wedge t$ and V, which is the product of their laws by independence. These measures agree on the set $\{[u,\infty) \times (v,\infty); u, v \in \mathbb{R}\}$, which is a π-system that generates $\mathcal{B}(\mathbb{R}^2)$. By Dynkin's Lemma, this implies that $P^{(A_t,B_t)} = P^{U\wedge t} \otimes P^V$. Therefore, for $u, v \geq 0$

$$P[A_t \geq u] = P^{(A_t,B_t)}[[u,\infty) \times \mathbb{R}] = P^{U\wedge t} \otimes P^V[[u,\infty) \times \mathbb{R}] = e^{-\lambda u}1_{\{u\leq t\}}$$

and

$$P[B_t > v] = P^{(A_t,B_t)}[\mathbb{R} \times (v,\infty)] = P^{U\wedge t} \otimes P^V[\mathbb{R} \times (v,\infty)] = e^{-\lambda v}.$$

This shows that $A_t \sim T_1 \wedge t$ and that $B_t \sim T_1$. We can also see from the steps above that for any $u, v \in \mathbb{R}$,

$$P[A_t \geq u, B_t > v] = P[A_t \geq u]P[B_t > v].$$

Since the families $\{[u,\infty); u \in \mathbb{R}\}$ and $\{(v,\infty); v \in \mathbb{R}\}$ are π-systems that generate $\mathcal{B}(\mathbb{R})$, we conclude using Dynkin's Lemma that A_t and B_t are independent.

(b) From (a) we know that the densities of A_t and B_t are given by

$$f_{A_t}(x) = 1_{\{0\leq x<t\}}\lambda e^{-\lambda x} + e^{-\lambda t}\delta_{(x,t)}, \quad f_{B_t}(x) = 1_{\{x\geq 0\}}\lambda e^{-\lambda x}.$$

Since A_t and B_t are independent, the density of L_t, is given by the convolution of f_{A_t} and f_{B_t}:

$$f_{L_t}(x) = \int_{\mathbb{R}} f_{A_t}(x-y)f_{B_t}(y)dy.$$

For $0 \leq x < t$:

$$f_{L_t}(x) = \int_0^x \lambda e^{-\lambda(x-y)}\lambda e^{-\lambda y}dy = \lambda^2 x e^{-\lambda x}\,.$$

For $x \geq t$:

$$f_{L_t}(x) = \int_{x-t}^x \lambda e^{-\lambda(x-y)}\lambda e^{-\lambda(y)}dy + e^{-\lambda t}\lambda e^{-\lambda(x-t)} = \lambda(1+\lambda t)e^{-\lambda x}$$

Hence,

$$E[L_t] = \int_0^\infty x f_{L_t}(x)dx = \frac{2 - \exp(-\lambda t)}{\lambda}.$$

It follows that

$$\lim_{t\to\infty} E[\beta_t] = \frac{2}{\lambda} = 2E[T_1].$$

We discover that the interval in which t falls is not a "typical" interval. To give a short explanation note that the probability of $t > 0$ lying in a large interval is larger than the probability of t being contained in a short interval. This bias causes the selected interval to be on the average twice as long as a typical interval.

Solution 2.2

(a) Let us denote φ_X the characteristic function of X_1. For every $s \in \mathbb{R}$ we have that

$$\begin{aligned}\varphi_{Z_t}(s) = \mathrm{E}[\exp(isZ_t)] &= \mathrm{E}\left[\exp\left(is\sum_{k=1}^{N_t} X_k\right)\right] = \mathrm{E}\left[\sum_{j=0}^{\infty}\exp\left(is\sum_{k=1}^{j} X_k\right)1_{\{N_t=j\}}\right] \\ &\overset{(1)}{=} \sum_{j=0}^{\infty}\mathrm{E}\left[\exp\left(is\sum_{k=1}^{j}X_k\right)\right]\cdot \mathrm{P}[N_t=j] \overset{(2)}{=} \sum_{j=0}^{\infty}\varphi_X(s)^j\cdot\frac{e^{-\lambda t}(\lambda t)^j}{j!} \\ &= \exp(\lambda t(\varphi_X(s)-1)).\end{aligned}$$

In (1) we used the dominated convergence theorem and independence between N_t and the X_i's. In (2) we used independence of the X_i's and that $N_t \sim \text{Poisson}(\lambda t)$. We also used the convention that empty sums are equal to 0.

(b) Note that for every $n \geq 2$, $0 = t_0 < t_1 < \cdots < t_n < \infty$ and $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$

$$\mathrm{P}[Z_{t_1}-Z_{t_0}\in A_1,\dots,Z_{t_n}-Z_{t_{n-1}}\in A_n] = \mathrm{P}\left[\sum_{k=N_{t_0}+1}^{N_{t_1}} X_k\in A_1,\dots,\sum_{k=N_{t_{n-1}}+1}^{N_{t_n}}X_k\in A_n\right]$$

$$= \sum_{(i_1,\dots,i_n)\in\mathbb{N}^n}\mathrm{P}\left[\sum_{k=1}^{i_1}X_k\in A_1,\dots,\sum_{k=i_1+\cdots+i_{n-1}+1}^{i_1+\cdots i_n}X_k\in A_n\right]\mathrm{P}[N_{t_1}=i_1,\dots,N_{t_n}-N_{t_{n-1}}=i_n] \tag{2}$$

where we used that $(N_t)_{t\geq 0}$ is independent of the X_i's. Since the increments of a Poisson process are stationary, we have that for $h > 0$

$$\mathrm{P}[N_{t_1}=i_1,\dots,N_{t_n}-N_{t_{n-1}}=i_n] = \mathrm{P}[N_{t_1+h}-N_h=i_1,\dots,N_{t_n+h}-N_{t_{n-1}+h}=i_n].$$

Then, replacing this in (2), and coming back through the same steps, we obtain

$$\mathrm{P}[Z_{t_1}-Z_{t_0}\in A_1,\dots,Z_{t_n}-Z_{t_{n-1}}\in A_n] = \mathrm{P}[Z_{t_1+h}-Z_{t_0+h}\in A_1,\dots,Z_{t_n+h}-Z_{t_{n-1}+h}\in A_n]$$

i.e., $(Z_{t_1}-Z_{t_0},\dots,Z_{t_n}-Z_{t_{n-1}}) \overset{(d)}{=} (Z_{t_1+h}-Z_{t_0+h},\dots,Z_{t_n+h}-Z_{t_{n-1}+h})$, and the process Z has stationary increments. If now we use the fact that the increments of the Poisson process $(N_t)_{t\geq 0}$ are independent, and that the random variables X_i's are also independent, we have that (2) equals to

$$\begin{aligned}\prod_{j=1}^{n}\sum_{i_j=1}^{\infty}\mathrm{P}\left[\sum_{k=i_1+\cdots+i_{j-1}+1}^{i_1+\cdots i_j}X_k\in A_j\right]\mathrm{P}[N_{t_j}-N_{t_{j-1}}=i_j] &= \prod_{j=1}^{n}\mathrm{P}\left[\sum_{k=N_{t_{j-1}}+1}^{N_{t_j}}X_k\in A_j\right] \\ &= \prod_{j=1}^{n}\mathrm{P}[Z_{t_j}-Z_{t_{j-1}}\in A_j]\end{aligned}$$

which means that Z has independent increments.

(c) If $X_i \sim \text{Bernoulli}(p)$ we have that $\varphi_X(s) = 1 + p(e^{is}-1)$. Hence, using part (a), we have that $\varphi_{Z_t}(s) = \exp(\lambda pt(e^{is}-1))$ and therefore $Z_t \sim \text{Poisson}(\lambda pt)$. Note that the process Z has jumps of size 1. Since it also has independent and stationary increments, we can conclude it is a Poisson process of rate λp.

Solution 2.3 We will start using the hint to prove (1). We know that the only non trivial σ-algebra we can consider in $\mathbb{N}^k$ is the power set $\mathcal{P}(\mathbb{N}^k)$ (in particular $\mathcal{B}(\mathbb{N}^k) = \mathcal{P}(\mathbb{N}^k)$). Then, any bounded function f is measurable and we can proceed by the so called *measure theoretic induction*. First, we want to prove that for any $A \in \mathcal{B}(\mathbb{N}^k)$, the identity (1) holds for $f = 1_A$, that is

$$\mathrm{P}[(N_{t_1}^{(n)}, \dots, N_{t_k}^{(n)}) \in A] \xrightarrow{n\to\infty} \mathrm{P}[(N_{t_1}, \dots, N_{t_k}) \in A].$$

From the hint, we know that this is true for the family $\{(i_1, \dots, i_k) \in \mathbb{N}^k\}$ which is a π-system which generates $\mathcal{B}(\mathbb{N}^k)$. Therefore, it is true for any $A \in \mathcal{B}(\mathbb{N}^k)$. Since the expectation is linear, (1) is also true for step functions of the type $f = \sum_{i=1}^k c_i 1_{A_i}$, where $c_i \in \mathbb{R}$ and $A_i \in \mathcal{B}(\mathbb{N}^k)$ for $i \in \{1, \dots, k\}$. If we consider a bounded function f, we know there exist a sequence of step functions bounded by $\|f\|_\infty$ and that converges pointwise to f. By the dominated convergence theorem, we can conclude that (1) holds for any bounded function f.

Let us now prove the hint. As in the previous exercises, we are going to take advantage of the independent increments of Poisson processes. We will also use that the sum of m independent random variables with distribution Bernoulli(p) has distribution Binomial(m, p).

Note that, if we define $t_0 = 0$ and $i_0 = 0$,

$$\begin{aligned}
&\mathrm{P}[N_{t_1}^{(n)} = i_1, \dots, N_{t_k}^{(n)} = i_k] = \mathrm{P}[N_{t_1}^{(n)} - N_{t_0}^{(n)} = i_1 - i_0, \dots, N_{t_k}^{(n)} - N_{t_{k-1}}^{(n)} = i_k - i_{k-1}] \\
&= \prod_{j=1}^k \mathrm{P}[N_{t_j}^{(n)} - N_{t_{j-1}}^{(n)} = i_j - i_{j-1}] \\
&= \prod_{j=1}^k \mathrm{P}\left[\sum_{i=\lfloor nt_{j-1}\rfloor+1}^{\lfloor nt_j \rfloor} X_i^{(n)} = i_j - i_{j-1}\right] \\
&= \prod_{j=1}^k \binom{\lfloor nt_j\rfloor - \lfloor nt_{j-1}\rfloor}{i_j - i_{j-1}} \left(1 - \frac{\lambda}{n}\right)^{\lfloor nt_j\rfloor - \lfloor nt_{j-1}\rfloor - (i_j - i_{j-1})} \left(\frac{\lambda}{n}\right)^{\lfloor nt_j\rfloor - \lfloor nt_{j-1}\rfloor}
\end{aligned}$$

Since $\frac{\lfloor nt_j\rfloor - \lfloor nt_{j-1}\rfloor}{n} \to t_j - t_{j-1}$ as $n \to \infty$, it follows that.

$$\mathrm{P}[N_{t_1}^{(n)} = i_1, \dots, N_{t_k}^{(n)} = i_k] \xrightarrow{n\to\infty} \prod_{j=1}^k \frac{e^{-\lambda(t_j - t_{j-1})}(\lambda(t_j - t_{j-1}))^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}$$

and since

$$\begin{aligned}
\mathrm{P}[N_{t_1} = i_1, \dots, N_{t_k} = i_k] &= \mathrm{P}[N_{t_1} - N_{t_0} = i_1 - i_0, \dots, N_{t_k} - N_{t_{k-1}} = i_k - i_{k-1}] \\
&= \prod_{j=1}^k \mathrm{P}[N_{t_j} - N_{t_{j-1}} = i_j - i_{j-1}] \\
&= \prod_{j=1}^k \frac{e^{-\lambda(t_j - t_{j-1})}(\lambda(t_j - t_{j-1}))^{i_j - i_{j-1}}}{(i_j - i_{j-1})!}
\end{aligned}$$

the hint is proven.