Applied Stochastic Processes

Exercise sheet 3

Exercise 3.1 Largest gap in a Poisson process.
Let \((N_t)_{t \geq 0}\) be a homogeneous Poisson process with parameter \(\lambda > 0\). The largest gap up to time \(t\) is defined as

\[ L_t = \max_{k \geq 1} (S_k \wedge t - S_{k-1} \wedge t). \]

In this exercise we are going to show that \(P\)-almost surely

\[ \limsup_{t \to \infty} \frac{L_t}{\log t} \leq \lambda - 1. \]

(a) Let \(\varepsilon > 0\). Use Borel-Cantelli’s lemma to show that \(P\)-almost surely

\[ \max_{1 \leq k \leq n} T_k \leq 1 + \varepsilon \log(n/\lambda) \]

for \(n\) large enough, where the \(T_k\) denote the inter-arrival times of the process.

(b) Show that \(P\)-almost surely

\[ N_t + 1 \leq (1 + \varepsilon)t\lambda \]

for large \(t\) enough.

(c) Conclude that \(P\)-almost surely \(\limsup_{t \to \infty} \frac{L_t}{\log t} \leq \lambda - 1\).

Exercise 3.2 Let \((N_t)_{t \geq 0}\) be a homogeneous Poisson process with rate \(\lambda > 0\). Let us define for every \(t \geq 0\), \(\tilde{N}_t := N_t + \lfloor t \rfloor\), where \(\lfloor \cdot \rfloor\) is the integer part function.

(a) Show that \((\tilde{N}_t)_{t \geq 0}\) has independent increments and that for all \(t \geq 0\) fixed

\[ P[\tilde{N}_{t+h} - \tilde{N}_t = 1] = \lambda h + o(h) \text{ and } P[\tilde{N}_{t+h} - \tilde{N}_t \geq 2] = o(h) \text{ as } h \to 0. \]

(b) Show that \((\tilde{N}_t)_{t \geq 0}\) is not an inhomogeneous Poisson process. Explain why this is not a contradiction with the infinitesimal characterization of inhomogeneous Poisson processes.

Exercise 3.3 Let \((N_t)_{t \geq 0}\) be an inhomogeneous Poisson process with \(N_0 = 0\) and rate \(\rho(t) = \alpha t\), where \(\alpha\) is a positive constant. Let

\[ S_n := \inf\{t > 0 : N_t = n\}, \quad n = 1, 2, \ldots \]

(a) Prove that for every \(k \geq 1\), \((S_1, \ldots, S_k)\) and \((R^{-1}(\tilde{T}_1), \ldots, R^{-1}(\tilde{T}_1 + \cdots + \tilde{T}_k))\) have the same distribution, where \((\tilde{T}_i)_{i \in \mathbb{N}}\) is an i.i.d. sequence of random variables with distribution \(\text{Exp}(1)\), and

\[ R(t) = \int_0^t \rho(s)ds \text{ for every } t \geq 0. \]

(b) Calculate the explicit joint distribution of \((S_1, S_2 - S_1, \ldots, S_k - S_{k-1})\) for every \(k \geq 2\). Conclude that the inter-arrival times of the process \((N_t)_{t \geq 0}\) are not independent.
Solution 3.1

(a) First we will show that P-a.s. there exists \( n_0 \) such that for all \( n \geq n_0 \) we have
\[
T_n \leq \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda).
\]
Set \( E_n := \{ T_n > \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda) \} \), then
\[
P[E_n] = \exp \left( -\lambda \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda) \right) = \left( \frac{\lambda}{n} \right)^{1+\varepsilon},
\]
hence \( \sum_n P[E_n] < \infty \) and therefore by Borel-Cantelli we obtain \( P[\lim \sup_{n \to \infty} E_n] = 0 \). This means that for P-a.a. \( \omega \) there is \( n_0(\omega) \) such that for all \( n \geq n_0(\omega) \) we have
\[
\max_{n_0(\omega) \leq k \leq n} T_k(\omega) \leq \frac{(1+\varepsilon)}{\lambda} \max_{n_0(\omega) \leq k \leq n} \log(k/\lambda) = \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda).
\]
Furthermore we can choose \( n_1(\omega) \geq n_0(\omega) \) such that
\[
\max_{1 \leq k \leq n_0(\omega)} T_k(\omega) \leq \frac{(1+\varepsilon)}{\lambda} \log(n_1(\omega)/\lambda),
\]
because \( \log \) is a monotone function increasing to infinity. Therefore P-a.s. there is \( n_1 \), such that for all \( n \geq n_1 \) we have
\[
\max_{1 \leq k \leq n} T_k(\omega) \leq \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda).
\]

(b) We have \( \lim \sup_{t \to \infty} \frac{N_{t+1}}{t} = \lim \sup_{t \to \infty} \frac{N_t}{t} \) and
\[
\lim \sup_{t \to \infty} \frac{N_t}{t} \leq \lim \sup_{t \to \infty} \frac{N_t}{S_{N_t}} = \lim \sup_{k \to \infty} \frac{k}{S_k} = \lambda,
\]
where we used in the last step that by the strong law of large numbers we have \( S_k/k \to \frac{1}{\lambda} \) almost surely as \( k \to \infty \). This implies that P-a.s. there is \( t_0 \) such that for all \( t > t_0 \) we have
\[
\frac{N_t+1}{t} \leq (1+\varepsilon)\lambda.
\]

(c) P-a.s. for \( t \) large enough we have
\[
L_t \leq \max_{1 \leq k \leq N_t+1} T_k \leq \frac{(1+\varepsilon)}{\lambda} \log \left( \frac{N_t+1}{\lambda} \right) \leq \frac{(1+\varepsilon)}{\lambda} \log(t(1+\varepsilon)),
\]
which yields \( \lim \sup_{t \to \infty} \frac{L_t}{\log t} \leq \frac{(1+\varepsilon)}{\lambda} \). As \( \varepsilon > 0 \) was arbitrarily chosen this yields the claim.

Solution 3.2

(a) Let \( 0 = t_0 < t_1 < \cdots < t_k < \infty \) and \( i_1, \ldots, i_k \in \mathbb{N} \), then
\[
P[\tilde{N}_{t_1} - N_{t_0} = i_1, \ldots, \tilde{N}_{t_k} - \tilde{N}_{t_{k-1}} = i_k]
= P[ N_{t_1} - N_{t_0} = i_1 + \lfloor t_1 \rfloor, \ldots, N_{t_k} - N_{t_{k-1}} = i_k + \lfloor t_k \rfloor - \lfloor t_{k-1} \rfloor - \lfloor t_k \rfloor]
= \prod_{j=1}^{k} P[ N_{t_j} - N_{t_{j-1}} = i_j + \lfloor t_{j-1} \rfloor - \lfloor t_j \rfloor]
= \prod_{j=1}^{k} P[ \tilde{N}_{t_j} - \tilde{N}_{t_{j-1}} = i_j],
\]
where we used the independence of the increments of \((N_t)_{t \geq 0}\). Therefore the independence of the increments of \((\tilde{N}_t)_{t \geq 0}\) is an easy consequence of Dynkin’s Lemma. On the other hand, notice that when \(h \to 0\) we have \(|t + h| - |t| = 0\). Then
\[
P[\tilde{N}_{t+h} - \tilde{N}_t = 1] = P[N_{t+h} - N_t = 1] = P[N_h = 1] = \lambda h + o(h) \text{ as } h \to 0
\]
(1)
because \((N_t)_{t \geq 0}\) is a Poisson process of rate \(\lambda\). In the same fashion, we have
\[
P[\tilde{N}_{t+h} - \tilde{N}_t \geq 2] = P[N_h \geq 2] = o(h) \text{ as } h \to 0.
\]

(b) If \((\tilde{N}_t)_{t \geq 0}\) were an inhomogeneous Poisson process, then from (1) we would have that \((\tilde{N}_t)_{t \geq 0}\) has constant rate, i.e. \((\tilde{N}_t)_{t \geq 0}\) would be a homogeneous Poisson process of rate \(\lambda\), which is not the case. Notice that the hypothesis of uniformity for bounded \(t\) is not satisfied for the process. To see that fix \(T > 0\) finite and \(h_0\) small and observe that \(|t + h| - |t|\) is not equal to 0 for every \(0 \leq t \leq T\) and \(0 \leq h \leq h_0\). Therefore, this is not a contradiction with the microscopic characterization of inhomogeneous Poisson processes.

**Solution 3.3**

(a) We know that there exists a homogeneous Poisson process \((\tilde{N}_t)_{t \geq 0}\) of rate 1, such that \(N_t = \tilde{N}_{R^{-1}(t)}\) for all \(t \geq 0\). Then
\[
S_k = \inf\{t > 0; N_t = k\} = \inf\{t > 0; \tilde{N}_{R^{-1}(t)} = k\} = R^{-1}(\inf\{t > 0; \tilde{N}_t = k\}) = R^{-1}(\tilde{S}_k),
\]
where \(\tilde{S}_k\) are the jumping times of the process \((\tilde{N}_t)_{t \geq 0}\). We can then write \(\tilde{S}_k = \tilde{T}_1 + \cdots + \tilde{T}_k\), where the \(\tilde{T}_i\)’s are i.i.d. random variables with distribution \(\text{Exp}(1)\). We conclude that \((S_1, \ldots, S_k) \overset{(d)}{=} (R^{-1}(\tilde{T}_1), \ldots, R^{-1}(\tilde{T}_1 + \cdots + \tilde{T}_k))\).

(b) Let us fix \(k \leq 2\) and denote \(S = (S_1, S_2 - S_1, \ldots, S_k - S_{k-1})\). Let us define the map \(\varphi : [0, \infty)^k \to [0, \infty)^k\) by
\[
\varphi(x_1, \ldots, x_k) = (R^{-1}(x_1), \ldots, R^{-1}(x_1 + \cdots + x_k) - R^{-1}(x_1 + \cdots + x_{k-1}))
\]
Then, we observe that \(\varphi(\tilde{T}_1, \ldots, \tilde{T}_k) \overset{(d)}{=} S\). Also note that
\[
\varphi^{-1}(y_1, \ldots, y_k) = (R(y_1), R(y_1 + y_2) - R(y_1), \ldots, R(y_1 + \cdots + y_k) - R(y_1 + \cdots + y_{k-1})).
\]
If \(f_{\tilde{T}}\) denotes the density of \((\tilde{T}_1, \ldots, \tilde{T}_k)\) and \(f_S\) the density of \(S\), we have by the transformation rule for densities, that for all \((y_1, \ldots, y_k) \in [0, \infty)^k\)
\[
f_S(y_1, \ldots, y_k) = f_{\tilde{T}}(\varphi^{-1}(y_1, \ldots, y_k)) \left| \frac{\partial \varphi}{\partial x} \right|_{1 \leq i, j \leq k} = f_{\tilde{T}}(\varphi^{-1}(y_1, \ldots, y_k)) \left| \frac{\partial \varphi}{\partial x} \right|_{1 \leq i, j \leq k}
\]
Note that the Jacobian matrix \(\varphi'\) is lower triangular, and then its determinant is given by the product of the elements of its diagonal. For \(R(t) = \int_0^t \alpha dt = \frac{1}{2} t^2\), we have \(R^{-1}(x) = \sqrt{2x} / \alpha\) and \(\frac{d}{dx} R^{-1}(x + y) = \frac{1}{\sqrt{2\alpha(x+y)}} = \frac{1}{\alpha R^{-1}(x+y)}\). Therefore
\[
\det \varphi'(\varphi^{-1}(y_1, \ldots, y_k)) = \prod_{i=1}^k \frac{1}{\alpha R^{-1}(R(y_1 + \cdots + y_i))} = \prod_{i=1}^k \frac{1}{\alpha(y_1 + \cdots + y_i)}.
\]
We also know that \( f_T(x_1, \ldots, x_k) = \prod_{i=1}^k e^{-x_i} 1\{x_i \geq 0\} = \exp(-(x_1 + \cdots + x_k)) 1\{x_1, \ldots, x_k \geq 0\} \).

It follows that

\[
\begin{align*}
  f_T(\varphi^{-1}(y_1, \ldots, y_k)) &= \exp(-R(y_1 + \cdots + y_k)) 1\{y_1, \ldots, y_k \geq 0\} \\
  &= \exp\left(-\frac{\alpha}{2} (y_1 + \cdots + y_k)^2\right) 1\{y_1, \ldots, y_k \geq 0\}
\end{align*}
\]

and then

\[
  f_S(y_1, \ldots, y_k) = \exp\left(-\frac{\alpha}{2} (y_1 + \cdots + y_k)^2\right) \prod_{i=1}^k \alpha(y_1 + \cdots + y_i) 1\{y_i \geq 0\}.
\]

A similar calculation shows that the previous density is not equal to the product

\[
  f_{S_1}(y_1) \cdots f_{S_k-S_k-1}(y_k),
\]

and therefore the inter-arrival times of the process \((N_t)_{t \geq 0}\) are not independent.