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Applied Stochastic Processes

Exercise sheet 3

Exercise 3.1 Largest gap in a Poisson process.
Let $(N_t)_{t\geq 0}$ be a homogeneous Poisson process with parameter $\lambda > 0$. The largest gap up to time t is defined as

$$L_t = \max_{k\geq 1}(S_k \wedge t - S_{k-1} \wedge t).$$

In this exercise we are going to show that P-almost surely

$$\limsup_{t\to\infty} \frac{L_t}{\log t} \leq \lambda^{-1}.$$

(a) Let $\varepsilon > 0$. Use Borel-Cantelli's lemma to show that P-almost surely

$$\max_{1\leq k\leq n} T_k \leq \frac{1+\varepsilon}{\lambda} \log(n/\lambda)$$

for n large enough, where the T_k denote the inter-arrival times of the process.

(b) Show that P-almost surely

$$N_t + 1 \leq (1+\varepsilon)t\lambda$$

for large t enough.

(c) Conclude that P-almost surely $\limsup_{t\to\infty} \frac{L_t}{\log t} \leq \lambda^{-1}$.

Exercise 3.2 Let $(N_t)_{t\geq 0}$ be a homogeneous Poisson process with rate $\lambda > 0$. Let us define for every $t \geq 0$, $\widetilde{N}_t := N_t + \lfloor t \rfloor$, where $\lfloor\cdot\rfloor$ is the integer part function.

(a) Show that $(\widetilde{N}_t)_{t\geq 0}$ has independent increments and that for all $t \geq 0$ fixed

$$\mathrm{P}[\widetilde{N}_{t+h} - \widetilde{N}_t = 1] = \lambda h + o(h) \text{ and } \mathrm{P}[\widetilde{N}_{t+h} - \widetilde{N}_t \geq 2] = o(h) \text{ as } h \to 0.$$

(b) Show that $(\widetilde{N}_t)_{t\geq 0}$ is not an inhomogeneous Poisson process. Explain why this is not a contradiction with the infinitesimal characterization of inhomogeneous Poisson processes.

Exercise 3.3 Let $(N_t)_{t\geqslant 0}$ be an inhomogeneous Poisson process with $N_0 = 0$ and rate $\rho(t) = \alpha t$, where α is a positive constant. Let

$$S_n := \inf\{t > 0 \ : \ N_t = n\}, \quad n = 1, 2, \ldots$$

(a) Prove that for every $k \geq 1$, $(S_1, \ldots, S_k)$ and $(R^{-1}(\widetilde{T}_1), \ldots, R^{-1}(\widetilde{T}_1 + \cdots + \widetilde{T}_k))$ have the same distribution, where $(\widetilde{T}_i)_{i\in\mathbb{N}}$ is an i.i.d. sequence of random variables with distribution Exp(1), and

$$R(t) = \int_0^t \rho(s)ds \text{ for every } t \geq 0.$$

(b) Calculate the explicit joint distribution of $(S_1, S_2 - S_1, \ldots, S_k - S_{k-1})$ for every $k \geq 2$. Conclude that the inter-arrival times of the process $(N_t)_{t\geq 0}$ are not independent.

Solution 3.1

(a) First we will show that P-a.s. there exists n_0 such that for all $n \geq n_0$ we have

$$T_n \leq \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda).$$

Set $E_n := \{T_n > \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda)\}$, then

$$\mathrm{P}[E_n] = \exp\left(-\lambda \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda)\right) = \left(\frac{\lambda}{n}\right)^{1+\varepsilon},$$

hence $\sum_n \mathrm{P}[E_n] < \infty$ and therefore by Borel-Cantelli we obtain $\mathrm{P}[\limsup_{n\to\infty} E_n] = 0$. This means that for P-a.a. ω there is $n_0(\omega)$ such that for all $n \geq n_0(\omega)$ we have

$$\max_{n_0(\omega)\leq k\leq n} T_k(\omega) \leq \frac{(1+\varepsilon)}{\lambda} \max_{n_0(\omega)\leq k\leq n} \log(k/\lambda) = \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda).$$

Furthermore we can choose $n_1(\omega) \geq n_0(\omega)$ such that

$$\max_{1\leq k\leq n_0(\omega)} T_k(\omega) \leq \frac{(1+\varepsilon)}{\lambda} \log(n_1(\omega)/\lambda),$$

because log is a monotone function increasing to infinity. Therefore P-a.s. there is n_1, such that for all $n \geq n_1$ we have

$$\max_{1\leq k\leq n} T_k(\omega) \leq \frac{(1+\varepsilon)}{\lambda} \log(n/\lambda).$$

(b) We have $\limsup_{t\to\infty} \frac{N_t+1}{t} = \limsup_{t\to\infty} \frac{N_t}{t}$ and

$$\limsup_{t\to\infty} \frac{N_t}{t} \leq \limsup_{t\to\infty} \frac{N_t}{S_{N_t}} = \limsup_{k\to\infty} \frac{k}{S_k} = \lambda,$$

where we used in the last step that by the strong law of large numbers we have $S_k/k \to \frac{1}{\lambda}$ almost surely as $k \to \infty$. This implies that P-a.s. there is t_0 such that for all $t > t_0$ we have

$$\frac{N_t+1}{t} \leq (1+\varepsilon)\lambda.$$

(c) P-a.s. for t large enough we have

$$L_t \leq \max_{1\leq k\leq N_t+1} T_k \leq \frac{(1+\varepsilon)}{\lambda} \log\left(\frac{N_t+1}{\lambda}\right) \leq \frac{(1+\varepsilon)}{\lambda} \log(t(1+\varepsilon)),$$

which yields $\limsup_{t\to\infty} \frac{L_t}{\log t} \leq \frac{(1+\varepsilon)}{\lambda}$. As $\varepsilon > 0$ was arbitrarily chosen this yields the claim.

Solution 3.2

(a) Let $0 = t_0 < t_1 < \cdots < t_k < \infty$ and $i_1, \ldots, i_k \in \mathbb{N}$, then

$$\begin{aligned}
&\mathrm{P}[\widetilde{N}_{t_1} - \widetilde{N}_{t_0} = i_1, \ldots, \widetilde{N}_{t_k} - \widetilde{N}_{t_{k-1}} = i_k] \\
&= \mathrm{P}[N_{t_1} - N_{t_0} = i_1 + \lfloor t_1 \rfloor, \ldots, N_{t_k} - N_{t_{k-1}} = i_k + \lfloor t_{k-1} \rfloor - \lfloor t_k \rfloor] \\
&= \prod_{j=1}^{k} \mathrm{P}[N_{t_j} - N_{t_{j-1}} = i_j + \lfloor t_{j-1} \rfloor - \lfloor t_j \rfloor] \\
&= \prod_{j=1}^{k} \mathrm{P}[\widetilde{N}_{t_j} - \widetilde{N}_{t_{j-1}} = i_j],
\end{aligned}$$

where we used the independence of the increments of $(N_t)_{t\geq 0}$. Therefore the independence of the increments of $(\widetilde{N}_t)_{t\geq 0}$ is an easy consequence of Dynkin's Lemma. On the other hand, notice that when $h \to 0$ we have $\lfloor t+h \rfloor - \lfloor t \rfloor = 0$. Then

$$\mathrm{P}[\widetilde{N}_{t+h} - \widetilde{N}_t = 1] = \mathrm{P}[N_{t+h} - N_t = 1] = \mathrm{P}[N_h = 1] = \lambda h + o(h) \text{ as } h \to 0 \tag{1}$$

because $(N_t)_{t\geq 0}$ is a Poisson process of rate λ. In the same fashion, we have

$$\mathrm{P}[\widetilde{N}_{t+h} - \widetilde{N}_t \geq 2] = \mathrm{P}[N_h \geq 2] = o(h) \text{ as } h \to 0.$$

(b) If $(\widetilde{N}_t)_{t\geq 0}$ were a inhomogeneous Poisson process, then from (1) we would have that $(\widetilde{N}_t)_{t\geq 0}$ has constant rate, i.e. $(\widetilde{N}_t)_{t\geq 0}$ would be a homogeneous Poisson process of rate λ, which is not the case. Notice that the hypothesis of uniformity for bounded t is not satisfied for the process. To see that fix $T > 0$ finite and h_0 small and observe that $\lfloor t+h \rfloor - \lfloor t \rfloor$ is not equal to 0 for every $0 \leq t \leq T$ and $0 \leq h \leq h_0$. Therefore, this is not a contradiction with the microscopic characterization of inhomogeneous Poisson processes.

Solution 3.3

(a) We know that there exists a homogeneous Poisson process $(\widetilde{N}_t)_{t\geq 0}$ of rate 1, such that $N_t = \widetilde{N}_{R^{-1}(t)}$ for all $t \geq 0$. Then

$$\begin{aligned} S_k &= \inf\{t > 0; N_t = k\} = \inf\{t > 0; \widetilde{N}_{R^{-1}(t)} = k\} \\ &= R^{-1}(\inf\{t > 0; \widetilde{N}_t = k\}) = R^{-1}(\widetilde{S}_k), \end{aligned}$$

where $\widetilde{S}_k$ are the jumping times of the process $(\widetilde{N}_t)_{t\geq 0}$. We can then write $\widetilde{S}_k = \widetilde{T}_1 + \cdots + \widetilde{T}_k$, where the $\widetilde{T}_i$'s are i.i.d. random variables with distribution Exp(1). We conclude that $(S_1, \ldots, S_k) \overset{(d)}{=} (R^{-1}(\widetilde{T}_1), \ldots, R^{-1}(\widetilde{T}_1 + \cdots + \widetilde{T}_k))$.

(b) Let us fix $k \leq 2$ and denote $S = (S_1, S_2 - S_1, \ldots, S_k - S_{k-1})$. Let us define the map $\varphi : [0,\infty)^k \to [0,\infty)^k$ by

$$\varphi(x_1, \ldots, x_k) = \big(R^{-1}(x_1), \ldots, R^{-1}(x_1 + \cdots + x_k) - R^{-1}(x_1 + \cdots + x_{k-1})\big)$$

Then, we observe that $\varphi(\widetilde{T}_1, \ldots, \widetilde{T}_k) \overset{(d)}{=} S$. Also note that

$$\varphi^{-1}(y_1, \ldots, y_k) = (R(y_1), R(y_1 + y_2) - R(y_1), \ldots, R(y_1 + \ldots + y_k) - R(y_1 + \ldots + y_{k-1})).$$

If $f_{\widetilde{T}}$ denotes the density of $(\widetilde{T}_1, \ldots, \widetilde{T}_k)$ and f_S the density of S, we have by the transformation rule for densities, that for all $(y_1, \ldots, y_k) \in [0,\infty)^k$

$$f_S(y_1, \ldots, y_k) = \frac{f_{\widetilde{T}}(\varphi^{-1}(y_1, \ldots, y_k))}{|\det \varphi'(\varphi^{-1}(y_1, \ldots, y_k))|} \text{ where } \varphi' = \left(\frac{\partial \varphi_i}{\partial x_j}\right)_{1\leq i,j\leq k}.$$

Note that the Jacobian matrix φ' is lower triangular, and then its determinant is given by the product of the elements of its diagonal. For $R(t) = \int_0^t \alpha t = \frac{\alpha}{2}t^2$, we have $R^{-1}(x) = \sqrt{\frac{2}{a}x}$ and $\frac{d}{dx}R^{-1}(x+y) = \frac{1}{\sqrt{2\alpha(x+y)}} = \frac{1}{\alpha R^{-1}(x+y)}$. Therefore

$$\det \varphi'(\varphi^{-1}(y_1, \ldots, y_k)) = \prod_{i=1}^k \frac{1}{\alpha R^{-1}(R(y_1 + \cdots + y_i))} = \prod_{i=1}^k \frac{1}{\alpha(y_1 + \cdots + y_i)}.$$

We also know that $f_{\widetilde{T}}(x_1, \ldots, x_k) = \prod_{i=1}^k e^{-x_i} 1_{\{x_i \geq 0\}} = \exp(-(x_1 + \cdots + x_k)) 1_{\{x_1,\ldots,x_k \geq 0\}}$. It follows that

$$\begin{aligned} f_T(\varphi^{-1}(y_1, \ldots, y_k)) &= \exp(-R(y_1 + \ldots + y_k)) 1_{\{y_1,\ldots,y_k \geq 0\}} \\ &= \exp\left(-\frac{\alpha}{2}(y_1 + \ldots + y_k)^2\right) 1_{\{y_1,\ldots,y_k \geq 0\}} \end{aligned}$$

and then

$$f_S(y_1, \ldots, y_k) = \exp\left(-\frac{\alpha}{2}(y_1 + \ldots + y_k)^2\right) \prod_{i=1}^k \alpha(y_1 + \cdots + y_i) 1_{\{y_i \geq 0\}}.$$

A similar calculation shows that the previous density is not equal to the product

$$f_{S_1}(y_1) \cdot \cdots \cdot f_{S_k - S_k - 1}(y_k),$$

and therefore the inter-arrival times of the process $(N_t)_{t \geq 0}$ are not independent.