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Applied Stochastic Processes

Exercise sheet 4

Exercise 4.1 Campbell's formula
Let N be a point process on $(E, \mathcal{E})$ with intensity measure μ, where μ is s-finite. Let $u : E \to \mathbb{R}$ be a measurable function. Show that $\int u(x)N(dx)$ is a well defined random variable and that if we have $u \geq 0$ or $\int |u(x)|\mu(dx) < \infty$, then

$$\mathrm{E}\left[\int u(x)N(dx)\right] = \int u(x)\mu(dx).$$

Exercise 4.2 Let N be a Poisson point process on $\mathbb{R}$ with intensity measure $\mu = \lambda \cdot \mathrm{Leb}(\mathbb{R})$, where $\lambda > 0$ and $\mathrm{Leb}(\mathbb{R})$ is the Lebesgue measure on $\mathbb{R}$. Let us order the points in $(0, \infty)$ as $0 < X_1 < X_2 < \cdots$.

(a) Show that $(X_n)_{n\geq 1}$ are well defined random variables.

(b) Prove that the random variables

$$Y_1 = X_1, \quad Y_n = X_n - X_{n-1} \text{ for } n \geq 2$$

are i.i.d. with distribution $\mathrm{Exp}(\lambda)$.

Exercise 4.3

(a) (Mapping Theorem) Let N be a Poisson process with s-finite intensity measure μ on a state space E with corresponding σ-algebra $\mathcal{E}$. Let $f : (E, \mathcal{E}) \to (F, \mathcal{F})$ be a measurable function and let $\mu^* = \mu \circ f^{-1}$ be the induced measure on F. Show that $N^* = N \circ f^{-1}$ is a Poisson process on F having intensity measure μ^*.

(b) Let N be a Poisson point process on $\mathbb{R}^d$ with intensity measure $\mu = \lambda \cdot \mathrm{Leb}(R^d)$, where $\lambda > 0$. Let B_r the ball of radius r around the origin. Prove that a.s.

$$\lim_{r\to\infty} \frac{N(B_r)}{|B_r|} = \lambda$$

where $|B_r|$ is the volume of B_r.
Hint: *Use the Mapping Theorem to study the process* $(N(B_r))_{r>0}$.

Solution 4.1 If $u(x) = 1_B(x)$ for some $B \in \mathcal{E}$, then $\int u(x)N(dx) = N(B)$. We know that $N(B) : \Omega \to \mathbb{N}_0$ is the mapping $\omega \mapsto N(\omega)(B)$, and it is a measurable map by definition. We also know that $\mathrm{E}[N(B)] = \mu(B)$. Then both assertions hold for this choice of u. By linearity we can extend this result to simple functions. Since the limit of measurable functions is measurable and using monotone convergence theorem we can also extend both assertions to arbitrary $u : E \to \mathbb{R}_0^+$. Now let us consider $u : E \to \mathbb{R}$. We can write $u = u_+ - u_-$ with $u_+, u_- : E \to \mathbb{R}_0^+$, and this implies that $\int u(x)N(dx)$ is a random variable. Assume that $\int |u(x)|\mu(dx) < \infty$. Then we have that

$$\begin{aligned}
\mathrm{E}\left[\int u(x)N(dx)\right] &= \mathrm{E}\left[\int u_+(x)N(dx)\right] - \mathrm{E}\left[\int u_-(x)N(dx)\right] \\
&= \int u_+(x)\mu(dx) - \int u_-(x)\mu(dx) \\
&= \int u(x)\mu(dx),
\end{aligned}$$

which concludes the proof.

Solution 4.2

(a) Let us prove that the process $(N_t)_{t\geq 0}$, defined by $N_0 = 0$ and $N_t = N((0,t])$ for $t \geq 0$, is a Poisson process with rate λ. We know that for every $\omega \in \Omega$, $N(\omega)$ is a s-finite measure, and then $\lim_{s\to t^+} N(\omega)((0,s]) = N(\omega)((0,t])$ which means that $N_t(\omega)$ is right continuous for every $\omega \in \Omega$. Since it has values in $\mathbb{N}_0$ we know it is a counting process. Let us consider $0 \leq s < t$. The fact that N is a random measure implies that $N_t - N_s = N((0,t]) - N((0,s]) = N((s,t])$ which is independent of $N((0,t]) = N_t$ and it has law Poisson$(\lambda(t-s))$. This concludes the proof.
Note that $X_n = \inf\{t > 0 : N((0,t]) = n\} = \inf\{t > 0 : N_t = n\}$ is well defined in the whole Ω and it is a measurable function.

(b) Since $(X_n)_{n\geq 1}$ coincides with the jumping times of the Poisson process of $(N_t)_{t\geq 0}$, we know that $(Y_n)_{n\geq 1}$ will be the inter-arrival times, which are i.i.d. random variables with distribution Exp(λ).

Solution 4.3

(a) Let us define $\mathcal{N}^*$ the space of s-finite measures on $(F, \mathcal{F})$. Set $\omega \in \Omega$. We know that $N(\omega) = \sum_n \nu_n$ where ν_n is a finite measure on $(E, \mathcal{E})$, for every $n \geq 1$. Then

$$N^*(\omega) = N(\omega) \circ f^{-1} = \sum_n \nu_n \circ f^{-1},$$

where for all $n \geq 1$ $\nu_n \circ f^{-1}$ is a finite measure on $(F, \mathcal{F})$, since f is measurable. It follows that $N^* : \Omega \to \mathcal{N}^*$. Let us show that this map is measurable. Let us consider $B \in \mathcal{F}$ and $k \in \mathbb{N}_0$. If we write $A = f^{-1}(B) \in \mathcal{E}$, we note that

$$\{N^* \in \{\nu \in \mathcal{N}^* : \nu(B) = k\}\} = \{N^*(B) = k\} = \{N(A) = k\} = \{N \in \{\eta \in \mathcal{N} : \eta(A) = k\}\}$$

and this last event is measurable, since N is a point process. We conclude that N^* is a point process on $(F, \mathcal{F})$. Let us prove that is a Poisson point process. Let us consider $B_1, \ldots, B_m$ disjoint sets in $\mathcal{F}$. Then, their pre-images $f^{-1}(B_1), \ldots, f^{-1}(B_m)$ are disjoint sets in $\mathcal{E}$. The independence of the random variables $N^*(B_1), \ldots, N^*(B_m)$ arise from the fact of N being a Poisson point process. In the same fashion, we have that $N^*(B_1) = N(f^{-1}(B_1)) \sim$ Poisson$(\mu(f^{-1}(B_1)) \sim$ Poisson$(\mu^*(B_1))$, and the statement follows.

(b) Let us define the map $f : \mathbb{R}^d \to \mathbb{R}_0^+$ by $f(x) = \|x\|_2 = \sqrt{x_1^2 + \cdots + x_d^2}$, which is a continuous function, an then measurable. By the mapping theorem, we know that $N^* = N \circ f^{-1}$ is a Poisson point process on R_0^+ with intensity measure $\mu^* = \mu \circ f^{-1}$. Let us define the process $(N_t^*)_{t\geq 0}$ by $N_t^* = N^*([0,t]))$. We use the fact that N^* is a random measure to see that for any $0 \leq s < t$

$$N_t^* - N_s^* = N^*([0,t]) - N^*([0,s]) = N^*((s,t])$$

which is independent of $N^*([0,s]) = N_s^*$. This shows that $(N_t^*)_{t\geq 0}$ has independent increments. We also have that

$$N_t^* - N_s^* = N^*((s,t]) = N(B_t \setminus B_s) \sim \text{Poisson}(\lambda(|B_t| - |B_s|)) \sim \text{Poisson}\left(\int_s^t \rho(x)dx\right)$$

with $\rho(x) = \lambda \pi_d \frac{x^{d-1}}{d}$, where π_d is the volume of the unit ball. This implies that $(N_t^*)_{t\geq 0}$ is a inhomogeneous Poisson process with rate ρ. By the time change property, we know that there exists a homogeneous Poisson process $(\widetilde{N}_t)_{t\geq 0}$ with rate 1 and such that $N_t^* = \widetilde{N}_{R(t)}$, where $R(t) = \int_0^t \rho(x)dx = \lambda \pi_d t^d = \lambda |B_t|$. Therefore

$$\frac{N(B_r)}{|B_r|} = \frac{N_r^*}{|B_r|} = \frac{\widetilde{N}_{R(r)}}{R(r)/\lambda}$$

Since R is continuous increasing, we can use the law of large numbers for Poisson processes to conclude that a.s. $\lim_{r\to\infty} \frac{N(B_r)}{|B_r|} = \lambda$.