Coordinator
Daniel Contreras Salinas

Applied Stochastic Processes

Exercise sheet 5

Exercise 5.1 Set $E := [0,1]$. We say that a set is *co-countable* if its complement is countable. Let $\mathcal{E}$ be the family of subsets of E that are either countable or co-countable.

(a) Show that $\mathcal{E}$ is a σ-algebra.

(b) Find a measure η on $(E,\mathcal{E})$ such that for all $B \in \mathcal{E}$, $\eta(B) \in \{0,1\}$, which is not of the form δ_x for some $x \in E$.

(c) Show that there exists a point process on $(E,\mathcal{E})$ which is not proper.

Exercise 5.2 Let N be a point process on $(E,\mathcal{E})$ with intensity measure μ and let $B \in \mathcal{E}$. Let $\mathcal{L}_N$ be the Laplace functional of N, which is given by

$$\mathcal{L}_N(u) = \mathrm{E}\left[\exp\left(-\int_E u(x)N(dx)\right)\right].$$

for all $u : E \to \mathbb{R}_+$ measurable.

(a) Show that if $\mu(B) < \infty$, then

$$\mu(B) = -\frac{d}{dt}\mathcal{L}_N(t1_B)\Big|_{t=0}.$$

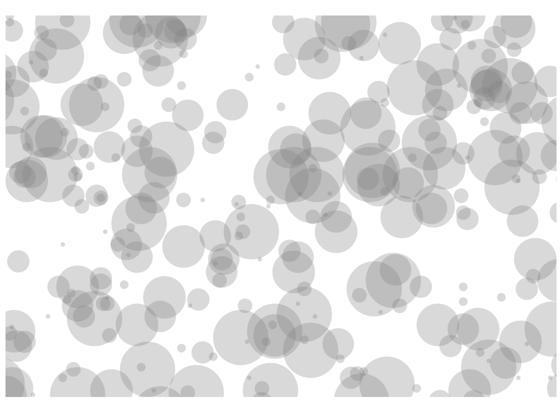
(b) We no longer assume that $\mu(B) < \infty$. Show that

$$\mathrm{P}[N(B) = 0] = \lim_{t\to\infty} \mathcal{L}_N(t1_B).$$

Exercise 5.3 Poisson Boolean model
Let $N = \sum_i \delta_{X_i}$ be a Poisson point process on $\mathbb{R}^d$ with intensity measure $\mu = \mathrm{Leb}(\mathbb{R}^d)$. Let us consider $(R_i)_i$ a sequence of i.i.d. positive random variables with law ρ, and independent of N. We define the *occupied* set by $\mathcal{O} = \bigcup_i B(X_i, R_i)$, where $B(x,r) \subset \mathbb{R}^d$ is the closed ball of center x and radius r.

(a) Let N_0 the number of balls $B(X_i, R_i)$ which contain the origin of $\mathbb{R}^d$. Show that N_0 is a well defined random variable with distribution Poisson $\left(\int_{\mathbb{R}^d}\int_{|x|}^{\infty} \rho(dr)\mu(dx)\right)$.

(b) Show that the event $\{\mathcal{O} = \mathbb{R}^d\}$ is measurable and that $\mathrm{P}[\mathcal{O} = \mathbb{R}^d] = 1$ if and only if $\int_0^\infty r^d \rho(dr) = \infty$.



Solution 5.1

(a) First we have $E \in \mathcal{E}$ since its complement is $\emptyset$. If $A \in \mathcal{E}$ is countable, then its complement is co-countable so $A^c \in \mathcal{E}$ and if A is co-countable, then its complement is countable so $A^c \in E$. Finally, to see that $\mathcal{E}$ is closed under countable unions, consider $(A_n)_{n\in\mathbb{N}} \subset \mathcal{E}$. Then either there is at least one A_i co-countable for some $i \in \mathbb{N}$, or all A_n's are countable. In the first case, we have

$$\left(\bigcup_{n\in\mathbb{N}} A_n\right)^c = \bigcap_{n\in\mathbb{N}} A_n^c \subset A_i^c,$$

which is countable. This implies that $\bigcup_{n\in\mathbb{N}} A_n$ is co-countable. In the second case, the countable union of countable sets is still countable.

(b) Define the mapping $\eta : \mathcal{E} \to \{0, 1\}$ such that for all $B \in \mathcal{E}$, $\eta(B) = 0$ if and only if B is countable (or equivalently $\eta(B) = 1 \Leftrightarrow B$ is co-countable). We need to check that η is indeed a measure. We clearly have $\eta(\emptyset) = 0$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint and measurable subsets in E. Suppose all A_n's are countable, then the countable union is also countable and $\eta(\bigcup_{n\in\mathbb{N}}) = 0 = \sum_{n\in\mathbb{N}} \eta(A_n)$. Now suppose there exists $i \neq j$ such that A_i, A_j are co-countable. Then $A_i \cap A_j = (A_i^c \cup A_j^c)^c \neq \emptyset$ because it is the complement of a countable subset. But then A_i and A_j are not disjoint, therefore there exists at most one co-countable subset in the sequence $(A_n)_{n\in\mathbb{N}}$. As in question (a), in that case we have $\bigcup_{n\in\mathbb{N}} A_n$ co-countable, which implies $\eta(\bigcup_{n\in\mathbb{N}} A_n) = 1$. We now note that $\sum_{n\in\mathbb{N}} \eta(A_n) = \eta(A_i) = 1$. The fact that η cannot be written as δ_x is proved assuming that there exists such $x \in E$, but then $\eta(\{x\}) = \delta_x(\{x\}) = 1$ which is in contradiction with the definition of η.

(c) We define $N(\omega) = \eta$ for every $\omega \in \Omega$. If N was proper we would have $N = \sum_i \delta_{X_i}$ where $X_i : \Omega \to E$ are measurable. Let us fix $\omega \in \Omega$ and a countable set $A \subset E$. Let us consider the set $B = \{X_1(\omega)\} \cup A$. We see that B is countable and then $\eta(B) = 0$. However, we also have that $N(\omega)(B) \geq 1$, which is a contradiction. Therefore N is a not proper Poisson point process.

Solution 5.2

(a) Recall by definition that $\mu(B) = \mathrm{E}[N(B)]$. We have

$$\mathcal{L}_N(t1_B) = \mathrm{E}\left[\exp\left(-t\int_E 1_B N(dx)\right)\right] = \mathrm{E}[\exp(-tN(B))].$$

Since $N(B) \geq 0$, the exponential above is bounded by 1. Besides, $N(B) \in L^1(\mathrm{P})$, so we can exchange the derivative and the expectation in the Laplace functional, therefore

$$-\frac{d}{dt}\mathcal{L}_N(t1_B) = \mathrm{E}[N(B)\exp(-tN(B))].$$

It suffices now to take $t = 0$ to conclude.

(b) For all $t > 0$, we have $L_N(t1_B) = \mathrm{E}[\exp(-tN(B))] = \mathrm{E}[1_{\{N(B)=0\}} + e^{-tN(B)}1_{\{N(B)\geq 1\}}]$ then by dominated convergence we get

$$\lim_{t\to\infty} \mathcal{L}_N(t1_B) = \mathrm{E}[1_{\{N(B)=0\}}] + 0 = \mathrm{P}[N(B) = 0].$$

Solution 5.3

(a) Let us consider the marked process $M = \sum_i \delta_{(X_i,R_i)}$. By the marking theorem, this is a Poisson point process on $\mathbb{R}^d \times \mathbb{R}^+$ with intensity measure $\mu \otimes \rho$. In this process, each point

(x, r) of the space corresponds to the ball $B(x, r)$. Note that the number of balls that intersect the origin is given by the number of points in the set $A = \{(x, r) \in \mathbb{R}^d \times \mathbb{R}^+; |x| \leq r\}$. In other words $N_0 = M(A)$. This implies that N_0 is a well defined random variable and that $N_0 \sim \text{Poisson}((\mu \otimes \rho)(A))$. We know by Fubini's Theorem that

$$(\mu \otimes \rho)(A) = \int_{\mathbb{R}^d \times \mathbb{R}^+} 1_A(y)(\mu \otimes \rho)(dy) = \int_{\mathbb{R}^d} \int_{|x|}^{\infty} \rho(dr)\mu(dx),$$

which shows what we wanted.

(b) Measurability of $\{\mathcal{O} = \mathbb{R}^d\}$ will be updated later. On the other hand, we know by Fubini's Theorem that

$$\int_{\mathbb{R}^d} \int_{|x|}^{\infty} \rho(dr)\mu(dx) = \int_0^{\infty} \int_{B(0,r)} \mu(dx)\rho(dr) = \pi_d \int_0^{\infty} r^d \rho(dr)$$

Hence,

$$\mathrm{P}[0 \notin \mathcal{O}] = \mathrm{P}[N_0 = 0] = \exp\left(-\pi_d \int_0^{\infty} r^d \rho(dr)\right).$$

Suppose that $\mathrm{P}[\mathcal{O} = \mathbb{R}^d] = 1$. Then $P[0 \in \mathcal{O}] = 1$, and we deduce from the last expression that $\int_0^{\infty} r^d \rho(dr) = \infty$. To prove the converse, assume that $\int_0^{\infty} r^d \rho(dr) = \infty$. As a preliminary result we first show for any $n \in \mathbb{N}$ that

$$(\mu \otimes \rho)\left(\{(x, r) \in \mathbb{R}^d \times \mathbb{R}^+ : B(0, n) \subset B(x, r)\}\right) = \infty. \tag{1}$$

Since $B(0, n) \subset B(x, r)$ if and only if $r \geq |x| + n$, the left-hand side of equation (1) equals

$$\int_0^{\infty} \int_{\mathbb{R}^d} 1_{\{r \geq |x|+n\}} \mu(dx)\rho(dr) = \pi_d \int_n^{\infty} (r - n)^d \rho(dr).$$

This is bounded below by

$$\pi_d \int_{2n}^{\infty} \left(\frac{r}{2}\right)^d \rho(dr) = \pi_d 2^{-d} \int_0^{\infty} 1_{\{r \geq 2n\}} r^d \rho(dr),$$

proving (1). Since M is a Poisson point process with intensity $\mu \otimes \rho$, the ball $B(0, n)$ is almost surely covered even by infinitely many of the balls $B(X_i, R_i)$. Since n is arbitrary, it follows that $\mathrm{P}[\mathcal{O} = \mathbb{R}^d] = 1$.