Applied Stochastic Processes

Exercise sheet 6

Exercise 6.1 Show that a renewal process with renewal function $M(t) = ct$, $t \geq 0$ for some constant $c > 0$ is a Poisson process.

Hint: *The Laplace transform determines the distribution.*

Exercise 6.2 Vehicles of random lengths arrive at a gate. Let L_k denote the length of the k-th vehicle. We assume that the random variables L_k are i.i.d. with distribution $3 + \text{Geometric}(1/2)$. The first vehicle that arrives parks directly at the gate. The vehicles arriving afterwards queue behind, leaving a random distance to the vehicle parked in front of themselves. We assume that these distances are independent and uniformly distributed on $[0, 1]$.

(a) For $x \geq 0$, let N_x denote the number of vehicles parked at distance at most x from the gate. Compute $\lim_{x\to\infty} N_x/x$.

(b) Suppose that the k-th vehicle is carrying D_k people, where the random variable D_k is distributed as $1 + \text{Binomial}(2L_k, 1/2)$. For $x \geq 0$ let $\widetilde{N}_x$ denote the number of people inside the vehicles parked at distance at most x from the gate. Estimate $\widetilde{N}_x$ for x large enough.

Exercise 6.3 Central Limit Theorem for Renewal Processes

If $(N_t)_{t\geq 0}$ is a renewal process with inter-arrival times T_i, $i \geq 1$, not a.s. constant and such that $E[T_1^2] < \infty$, show that when $t \to \infty$,

$$Z_t := \frac{N_t - t/\mu}{\sigma(t/\mu^3)^{\frac{1}{2}}}$$

converges in law to the standard normal distribution, where $\mu = E[T_1]$ and $\sigma^2 = \text{Var}(T_1) > 0$.

Hint: *Let $S_n := T_1 + ... + T_n$, then by the central limit theorem*

$$\lim_{n\to\infty} P[(S_n - n\mu)/\sigma\sqrt{n} \leq x] = \Phi(x)$$

uniformly *in $x \in \mathbb{R}$, where Φ denotes the distribution function of the standard normal distribution.*

Solution 6.1 We know that the Laplace transform L_m of the renewal function m satisfies that $L_m(s) = \frac{L_F(s)}{1-L_F(s)}$ for every $s \geq 0$. Using

$$L_m(s) = \int_0^\infty \mathrm{e}^{-st} dm(t) = c\int_0^\infty \mathrm{e}^{-st} dt = \frac{c}{s},$$

we can derive that

$$L_F(s) = \frac{L_m(s)}{1+L_m(s)} = \frac{1}{(s/c)+1}.$$

On the other hand, the Laplace transform of an $\mathrm{Exp}(c)$ random variable is given by $\int_0^\infty e^{-st} ce^{-ct} dt = \frac{1}{(s/c)+1}$. Because the Laplace transform determines the distribution, the interarrival times are $\mathrm{Exp}(c)$ distributed. Therefore, We have a renewal process starting at 0 with jump size 1, whose interarrival time are exponentially distributed with the parameter c. We can then conclude that the renewal process is a Poisson process with rate c.

Solution 6.2

(a) For $k \in \mathbb{N}$ let U_k denote the distance between the $(k+1)$-st and the k-th vehicle queueing at the gate. Then for all $x \geq 0$,

$$N_x = 1 + \sum_{k=1}^{\infty} 1_{\left\{\sum_{j=1}^{k} L_j + U_j \leq x\right\}}.$$

Hence, $(N_x - 1)_{x\geq 0}$ is a renewal process with interarrival times $T_k = L_k + U_k$. Note that $E[T_k] = \frac{1}{2} + 4 < \infty$. The strong law of large numbers for renewal processes implies

$$\lim_{x\to\infty} \frac{N_x}{x} = \frac{2}{9} \quad \text{a.s.}$$

(b) Note that

$$\widetilde{N}_x = D_1 + \sum_{k=1}^{\infty} D_{k+1} 1_{\left\{\sum_{j=1}^{k} T_j \leq x\right\}}.$$

Then $(\widetilde{N}_x - D_1)_x$ is a renewal process with reward $(D_{i+1})_i$. Therefore, since $\mathrm{E}[D_1] = 1 + \mathrm{E}[L_k] = 5 < \infty$ we conclude that

$$\lim_{x\to\infty} \frac{\widetilde{N}_x}{x} = \frac{\mathrm{E}[D_i]}{\mathrm{E}[T_i]} = \frac{10}{9}.$$

Remark: we considered L_k with support in $\mathbb{N}_0$.

Solution 6.3 Let Φ denote the distribution function of the standard normal distribution, and let $\lfloor x \rfloor$ be the greatest integer less than or equal to x for $x \in \mathbb{R}$. Let $S_n := \sum_{i=1}^{n} T_i$, then using the central limit theorem we have

$$\lim_{n\to\infty} P[(S_n - n\mu)/\sigma\sqrt{n} \leq x] = \Phi(x)$$

uniformly in $x \in \mathbb{R}$.
Now, for given $t > 0$ and $x \in \mathbb{R}$, since N_t is integer-valued, we have

$$P[Z_t \leq x] = P\left[N_t \leq \lfloor x(\sigma(t/\mu^3)^{\frac{1}{2}}) + t/\mu \rfloor\right]. \tag{1}$$

Setting $h(t) := \lfloor x(\sigma(t/\mu^3)^{\frac{1}{2}}) + t/\mu \rfloor$, from

$$\{N_t \leq h(t)\} = \{S_{h(t)} \geq t\}$$

we obtain that

$$(1) = P[S_{h(t)} \geq t] = P\left[(S_{h(t)} - \mu h(t))/\sigma\sqrt{h(t)} \geq (t - \mu h(t))/\sigma\sqrt{h(t)}\right]. \tag{2}$$

It suffices to show $h(t) \to \infty$ and $z(t) := (t - \mu h(t))/\sigma\sqrt{h(t)} \to -x$ as $t \to \infty$, since in that case the *uniform convergence* in the central limit theorem will imply

$$P\left[(S_{h(t)} - \mu h(t))/\sigma\sqrt{h(t)} \geq z(t)\right] \to 1 - \Phi(-x) = \Phi(x),$$

which means that $P[Z_t \leq x] \to \Phi(x)$ and therefore Z_t converges to the standard normal distribution in law as $t \to \infty$. Indeed, if a sequence of functions $(f_n)_{n\geq 1}$ converges *uniformly* to a *continuous* function f, and a sequence of real numbers $(y_n)_{n\geq 1}$ converges to some $y \in \mathbb{R}$, then one can easily prove that $\lim_{n\to\infty} f_n(y_n) = f(y)$. Now for any sequence $(t_n)_{n\geq 1}$ tending to infinity, we can define f_n as the distribution function of $(S_{h(t_n)} - \mu h(t_n))/\sigma\sqrt{h(t_n)}$ and $y_n := z(t_n)$. Since f_n converges uniformly to the function $f(x) := 1 - \Phi(x)$ and y_n converges to $y := -x$, using the above claim we can deduce the desired result.
The fact that $\lim_{t\to\infty} h(t) = \infty$ is easy to see. To show that $\lim_{t\to\infty} z(t) = -x$, we first note that by definition $h(t) = x(\sigma(t/\mu^3)^{\frac{1}{2}}) + t/\mu + \epsilon(t)$, where $|\epsilon(t)| < 1$, and hence

$$\begin{aligned} z(t) &= \frac{t - \mu[x(\sigma(t/\mu^3)^{\frac{1}{2}}) + t/\mu + \epsilon(t)]}{\sigma\sqrt{h(t)}} \\ &\sim \frac{-\mu x(\sigma(t/\mu^3)^{\frac{1}{2}})}{\sigma\sqrt{t/\mu}} \\ &\to -x \text{ as } t \to \infty. \end{aligned}$$