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Applied Stochastic Processes

Exercise sheet 7

Exercise 7.1 Cycles of operation and repair of a machine.
Let $(U_i, V_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random vectors with $U_i \geq 0$, $V_i \geq 0$. Assume that $T_i = U_i + V_i$ is not almost surely equal to 0 and denote by F its distribution function. We interpret U_i and V_i as alternating periods when a given machine is operational or in repair. The period U_1 begins at time 0. For $t \geq 0$ we define $Y_t = 1$ if the machine is operational at time t and $Y_t = 0$ otherwise. Let $g(t) = P[Y_t = 1]$ denote the probability of the machine being operational at time $t \geq 0$, and $g(t) = 0$ for $t < 0$. We also define $h(t) = \mathrm{P}[U_1 > t]$.

(a) Prove that for $t \geq 0$

$$g(t) = h(t) + \int_0^t g(t-s)dF(s),$$

i.e. that g is the solution of the (h, F)-renewal equation.

(b) Assume that $\mathrm{E}[U_1] < \infty$ and that F is non-arithmetic. Show that the function h is directly Riemann integrable and conclude that

$$\lim_{t\to\infty} g(t) = \frac{\mathrm{E}[U_1]}{\mathrm{E}[U_1] + \mathrm{E}[V_1]}.$$

Exercise 7.2 Let N be a renewal process with renewal times $(S_k)_{k\geq 0}$, where $S_0 = 0$, and interarrival distribution F having finite mean $\mu > 0$. Denote by A the age process of N, i.e. $A_t = t - S_{N_t}$ for $t \geq 0$. For $x \geq 0$, set $\varphi_x(t) = \mathrm{P}[A_t \leq x]$ for $t \geq 0$, and $\varphi_x(t) = 0$ for $t < 0$.

(a) Show that φ_x satisfies the renewal equation

$$\varphi_x(t) = 1_{\{t\leq x\}}(1 - F(t)) + \int_0^t \varphi_x(t-s)dF(s) \text{ for } t \geq 0.$$

(b) Assume that F is non-arithmetic. Compute $\lim_{t\to\infty} \varphi_x(t)$. Deduce that A_t converges in distribution to some random variable A_∞ as $t \to \infty$.

Solution 7.1

(a) Let $S_k = T_1 + \ldots + T_k$. For $t \geq 0$,

$$\begin{aligned} g(t) &= \mathrm{P}[Y_t = 1] \\ &= \mathrm{P}[Y_t = 1, T_1 > t] + \mathrm{P}[Y_t = 1, T_1 \leq t] \\ &= \mathrm{P}[U_1 > t] + \mathrm{E}\left[\sum_{k\geq 0} 1_{\{S_k \leq t < S_k + U_{k+1}\}} 1_{\{T_1 \leq t\}}\right] \\ &= \mathrm{P}[U_1 > t] + \mathrm{E}\left[\sum_{k\geq 1} 1_{\{T_1 + S_k - S_1 \leq t < T_1 + S_k - S_1 + U_{k+1}\}}\right], \end{aligned}$$

where T_1 is independent of $S_k - S_1$ and of U_{k+1}, and $S_k - S_1 \overset{(d)}{=} S_{k-1}$ for $k \geq 1$. This implies that

$$\begin{aligned} g(t) &= \mathrm{P}[U_1 > t] + \int_0^t \mathrm{E}\left[\sum_{k\geq 1} 1_{\{S_{k-1} \leq t-s < S_{k-1} + U_{k+1}\}}\right] dF(s) \\ &= h(t) + \int_0^t g(t-s) dF(s) \end{aligned}$$

(b) Note that $h \geq 0$ and it is a non increasing function. Also

$$\int_0^\infty h(t)dt = \int_0^\infty \mathrm{P}[U_1 > t]dt = \mathrm{E}[U_1] < \infty,$$

which means that h is directly Riemann integrable. Since F is non-arithmetic and g is solution of the equation $g = h + g * F$, we know by Smith's key renewal theorem that

$$\lim_{t\to\infty} g(t) = \frac{1}{\mathrm{E}[T_1]}\mathrm{E}[U_1] = \frac{\mathrm{E}[U_1]}{\mathrm{E}[U_1] + \mathrm{E}[V_1]}.$$

Solution 7.2

(a) Set $t \geq 0$ and $x \geq 0$. Note that

$$\begin{aligned} \varphi_x(t) &= \mathrm{P}[A_t \leq x] = \mathrm{P}[A_t \leq x, S_1 > t] + P[A_t \leq x, S_1 \leq t] \\ &= 1_{\{t \leq x\}}(1 - F(t) + \mathrm{P}[t - S_{N_t} \leq x, S_1 \leq t]. \end{aligned}$$

We know that $\{S_1 \leq t\} = \{N_t \geq 1\}$. Then

$$\begin{aligned} \mathrm{P}[t - S_{N_t} \leq x, S_1 \leq t] &= \sum_{n\geq 1} \mathrm{P}[t - S_n \leq x, N_t = n] \\ &= \sum_{n\geq 1} \mathrm{P}[t - x \leq S_n \leq t < S_{n+1}] \end{aligned}$$

We can write $S_n = S_n - S_1 + T_1$, where T_1 is independent of $S_n - S_1$ and $S_n - S_1 \overset{(d)}{=} S_{n-1}$. Therefore

$$\begin{aligned} \mathrm{P}[t - S_{N_t} \leq x, S_1 \leq t] &= \sum_{n\geq 1} \int_0^t \mathrm{P}[t - x \leq s + S_{n-1} \leq t < s + S_n] dF(s) \\ &= \sum_{n\geq 1} \int_0^t \mathrm{P}[t - s - x \leq S_{n-1} \leq t - s < S_n] dF(s) \\ &= \sum_{n\geq 1} \int_0^t \mathrm{P}[A_{t-s} \leq x, N_{t-s} = n-1] dF(s) \\ &= \int_0^t \mathrm{P}[A_{t-s} \leq x] dF(s) = \int_0^t \varphi_x(t-s) dF(s), \end{aligned}$$

which is exactly what we wanted.

(b) Let us define $h(t) = 1_{\{t \leq x\}}(1 - F(t))$ for $t \geq 0$. Note that $h \geq 0$, it is measurable, continuous a.e. and bounded by 1. Also it vanishes outside the compact interval $[0, x]$. This implies that h is directly Riemann integrable. Since F is non-arithmetic, by Smith's key renewal theorem it follows that

$$\lim_{t \to \infty} \varphi_x(t) = \frac{1}{\mathrm{E}[T_1]} \int_0^\infty h(t)dt = \frac{1}{\mu} \int_0^x (1 - F(t))dt = G_*(x).$$

This means that A_t converges in distribution to a random variable with distribution G_* (recall that G_* is the distribution function which makes stationary the F-renewal with delay G_*).