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Applied Stochastic Processes

Exercise sheet 8

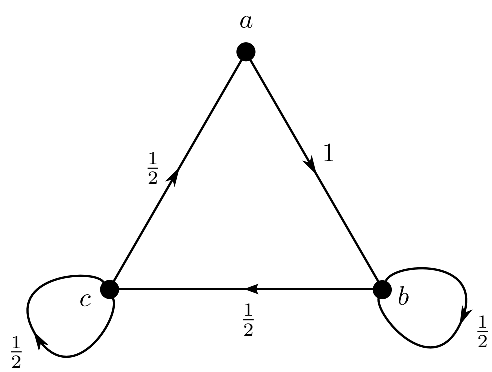
Exercise 8.1 A die is rolled repeatedly. Which of the following stochastic processes $(X_n)_{n\in\mathbb{N}}$ are Markov chains? For those that are, determine the transition probability and in (b), additionally, the n-step transition probability.

(a) Let X_n denote the number of rolls at time n since the most recent six.

(b) Let X_n denote the largest number that has come up in the first n rolls.

(c) Let X_n denote the larger number of those that came up in the rolls number $n-1$ and n (the last two rolls), and we consider $(X_n)_{n\geq 2}$.

Exercise 8.2 Consider the three-state Markov chain with initial distribution $\mu = \delta_a$ an transition probability given by the following diagram



Prove that

$$\mathbb{P}[X_n = a] = \frac{1}{5} + \left(\frac{1}{2}\right)^n \left(\frac{4}{5}\cos\frac{n\pi}{2} - \frac{2}{5}\sin\frac{n\pi}{2}\right).$$

Exercise 8.3 Let $\xi_1, \xi_2, \ldots$ be i.i.d. uniform random variables on the set $\{1, \ldots, N\}$.

(a) Show that $X_n = |\{\xi_1, \ldots, \xi_n\}|$ is a Markov chain and compute its transition probability.

(b) Compute $\mathbb{P}[X_n = i]$ for $n \geq 1$ and $i \in \{1, \ldots, N\}$.

Solution 8.1 The stochastic processes described in a) and b) are Markov chains, while the one in c) is not. Let Y_n denote the number which shows up in the n-th roll, which is independent of $X_1, \dots, X_{n-1}$.

(a) We have $X_n = (X_{n-1}+1)\,1_{\{Y_n<6\}}$. Thus, $(X_n)_{n\in\mathbb{N}}$ is a Markov chain with state space $\mathbb{N}_0$. For $i, j \in \{0,1,2,\dots\}$:

$$p_{i,j} = \begin{cases} \frac{1}{6} & \text{if } j = 0, \\ \frac{5}{6} & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Then $X_n = \max\{X_{n-1}, Y_n\}$. Hence, $(X_n)_{n\in\mathbb{N}}$ is a Markov chain with state space $\{1,\dots,6\}$. We obtain the following transition probabilities for $1 \le i, j \le 6$:

$$p_{i,j} = \begin{cases} 0 & \text{if } j < i, \\ \frac{i}{6} & \text{if } j = i, \\ \frac{1}{6} & \text{if } j > i. \end{cases}$$

Furthermore, noting that $p_{i,j}^{(n)} = P\big[\max\{Y_1, Y_2, \dots, Y_n\} = j \mid X_0 = i\big]$ for $j > i$, we have

$$p_{i,j}^{(n)} = \begin{cases} 0 & \text{if } j < i, \\ \left(\frac{i}{6}\right)^n & \text{if } j = i, \\ \left(\frac{j}{6}\right)^n - \left(\frac{j-1}{6}\right)^n & \text{if } j > i. \end{cases}$$

(c) The transition probabilities at time n depend not only on X_n, but also on X_{n-1}. For example,

$$\begin{aligned} \mathrm{P}[X_4 = 6 | X_3 = 6] = P[Y_3 = 6 | X_3 = 6] + \mathrm{P}[Y_3 < 6, Y_4 = 6 | X_3 = 6] = \frac{6}{11} + \frac{5}{11}\cdot\frac{1}{6} \\ < 1 = \mathrm{P}[X_4 = 6 | X_3 = 6, X_2 = 1]. \end{aligned}$$

Therefore, this is not a Markov chain.

Solution 8.2 Let us identify the set a, b, c with $1, 2, 3$. Then, from the diagram we can get the following transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

We know that $\mathbb{P}[X_n = a] = p_{a,a}^{(n)} = P^n(1,1)$. Then we need to calculate P^n. We see that this matrix is diagonalizable since it has different eigenvalues. Indeed, it characteristic equation is given by

$$0 = \det(\lambda I - P) = \lambda\left(\lambda - \frac{1}{2}\right)^2 - \frac{1}{4} = \frac{1}{4}(\lambda - 1)(4\lambda^2 + 1)$$

and its eigenvalues are $1, i/2, -i/2$. Hence, there exists an invertible matrix U such that

$$P = U\begin{pmatrix} 1 & 0 & 0 \\ 0 & i/2 & 0 \\ 0 & 0 & -i/2 \end{pmatrix}U^{-1}$$

and then

$$P^n = U\begin{pmatrix} 1 & 0 & 0 \\ 0 & (i/2)^n & 0 \\ 0 & 0 & (-i/2)^n \end{pmatrix}U^{-1}$$

This implies that $P^n(1,1) = x + y(i/2)^n + z(-i/2)^n$ for some constants x, y, z. We can calculate the value of these constants by using the first steps of our chain

$$\begin{aligned} 1 &= P^0(1,1) = x + y + z \\ 0 &= P^1(1,1) = x + iy/2 - iz/2 \\ 0 &= P^2(1,1) = x - y/4 - z/4. \end{aligned}$$

This give us $x = 1/5$, $y = (i-2)/5$ and $z = (2-i)/5$. Therefore

$$\begin{aligned} P^n(1,1) &= \frac{1}{5} + \frac{i-2}{5}\left(\frac{i}{2}\right)^n + \frac{2-i}{5}\left(\frac{-i}{2}\right)^n \\ &= \frac{1}{5} + \frac{i-2}{5}\left(\frac{1}{2}\right)^n\left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right) + \frac{2-i}{5}\left(\frac{1}{2}\right)^n\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right) \\ &= \frac{1}{5} + \left(\frac{1}{2}\right)^n\left(\frac{4}{5}\cos\frac{n\pi}{2} - \frac{2}{5}\sin\frac{n\pi}{2}\right). \end{aligned}$$

Solution 8.3

(a) This is a Markov chain since the probability of adding a new value at time $n+1$ depends on the number of values we have seen up to time n.

$$p_{i,j} = \begin{cases} \frac{N-i}{N} & \text{if } j = i+1, \\ \frac{i}{N} & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Since $\xi_1, \xi_2, \dots$ are i.i.d. uniform random variables on $\{1, \dots, N\}$, we have that

$$\begin{aligned} \mathrm{P}[X_n = i] &= \mathrm{P}[|\{\xi_1, \dots, \xi_n\}| = i] \\ &= \sum_{\substack{I \subset \{1,\dots,N\} \\ |J| = i}} \mathrm{P}[\{\xi_1, \dots, \xi_n\} = J] \\ &= \binom{N}{i} \mathrm{P}[\{\xi_1, \dots, \xi_n\} = \{1, \dots, i\}]. \end{aligned}$$

Let us call $\mathcal{X}_n = \{\xi_1, \dots, \xi_n\}$ and $I = \{1, \dots, i\}$, then

$$\mathrm{P}[\mathcal{X}_n = I] = \mathrm{P}[\mathcal{X}_n \subset I] - \mathrm{P}\left[\mathcal{X}_n \subset I, \bigcup_{k=1}^{i} \{k \notin \{\xi_1, \dots, \xi_n\}\}\right]$$

We know that $\mathrm{P}[1, \dots, k \notin \mathcal{X}_n, \mathcal{X}_n \subset I] = \mathrm{P}[\xi_1, \dots, \xi_n \in \{k+1, \dots, i\}] = \left(\frac{i-k}{N}\right)^n$. Since there are $\binom{i}{k}$ ways of choosing the elements that do not appear in $\mathcal{X}_n$ and using the Inclusion-Exclusion principle, we have that

$$\mathrm{P}\left[\mathcal{X}_n \subset I, \bigcup_{k=1}^{i} \{k \notin \{\xi_1, \dots, \xi_n\}\}\right] = \sum_{k=1}^{n} (-1)^{k-1} \binom{i}{k} \left(\frac{i-k}{N}\right)^n.$$

Since $\mathrm{P}[\mathcal{X}_n \subset I] = \left(\frac{i}{N}\right)^n$, we can put everything together to get

$$\mathrm{P}[X_n = i] = \binom{N}{i} \sum_{k=0}^{n} (-1)^k \binom{i}{k} \left(\frac{i-k}{N}\right)^n.$$