Applied Stochastic Processes

Exercise sheet 8

Exercise 8.1 A die is rolled repeatedly. Which of the following stochastic processes \((X_n)_{n \in \mathbb{N}}\) are Markov chains? For those that are, determine the transition probability and in (b), additionally, the \(n\)-step transition probability.

(a) Let \(X_n\) denote the number of rolls at time \(n\) since the most recent six.

(b) Let \(X_n\) denote the largest number that has come up in the first \(n\) rolls.

(c) Let \(X_n\) denote the larger number of those that came up in the rolls number \(n - 1\) and \(n\) (the last two rolls), and we consider \((X_n)_{n \geq 2}\).

Exercise 8.2 Consider the three-state Markov chain with initial distribution \(\mu = \delta_a\) an transition probability given by the following diagram

\[
\begin{array}{ccc}
    & 1 & \\
\frac{1}{2} & 1 & \frac{1}{2} \\
\frac{1}{2} & 1 & b
\end{array}
\]

Prove that
\[
P[X_n = a] = \frac{1}{5} + \left(\frac{1}{2}\right)^n \left(\frac{4}{5} \cos \frac{n\pi}{2} - \frac{2}{5} \sin \frac{n\pi}{2}\right).
\]

Exercise 8.3 Let \(\xi_1, \xi_2, \ldots\) be i.i.d. uniform random variables on the set \(\{1, \ldots, N\}\).

(a) Show that \(X_n = |\{\xi_1, \ldots, \xi_n\}|\) is a Markov chain and compute its transition probability.

(b) Compute \(P[X_n = i]\) for \(n \geq 1\) and \(i \in \{1, \ldots, N\}\).
**Solution 8.1** The stochastic processes described in a) and b) are Markov chains, while the one in c) is not. Let \( Y_n \) denote the number which shows up in the \( n \)-th roll, which is independent of \( X_1, \ldots, X_{n-1} \).

(a) We have \( X_n = (X_{n-1} + 1) 1_{\{Y_n < 6\}} \). Thus, \( (X_n)_{n \in \mathbb{N}} \) is a Markov chain with state space \( \mathbb{N}_0 \).

For \( i, j \in \{0, 1, 2, \ldots\} \):

\[
p_{i,j} = \begin{cases} 
\frac{1}{6} & \text{if } j = 0, \\
\frac{5}{6} & \text{if } j = i + 1, \\
0 & \text{otherwise}.
\end{cases}
\]

(b) Then \( X_n = \max\{X_{n-1}, Y_n\} \). Hence, \( (X_n)_{n \in \mathbb{N}} \) is a Markov chain with state space \( \{1, \ldots, 6\} \).

We obtain the following transition probabilities for \( 1 \leq i, j \leq 6 \):

\[
p_{i,j} = \begin{cases} 
0 & \text{if } j < i, \\
\left(\frac{i}{6}\right)^n & \text{if } j = i, \\
\left(\frac{j}{6}\right)^n - \left(\frac{j-1}{6}\right)^n & \text{if } j > i.
\end{cases}
\]

Furthermore, noting that \( p_{i,j}^{(n)} = P[\max\{Y_1, Y_2, \ldots, Y_n\} = j \mid X_0 = i] \) for \( j > i \), we have

\[
p_{i,j}^{(n)} = \begin{cases} 
0 & \text{if } j < i, \\
\left(\frac{j}{6}\right)^n & \text{if } j = i, \\
\left(\frac{j}{6}\right)^n - \left(\frac{j-1}{6}\right)^n & \text{if } j > i.
\end{cases}
\]

(c) The transition probabilities at time \( n \) depend not only on \( X_n \), but also on \( X_{n-1} \). For example,

\[
P[X_4 = 6 \mid X_3 = 6] = P[Y_3 = 6 \mid X_3 = 6] + P[Y_3 < 6, Y_4 = 6 \mid X_3 = 6] = \frac{6}{11} + \frac{5}{11} \cdot \frac{1}{6}
\]

\[
< 1 = P[X_4 = 6 \mid X_3 = 6, X_2 = 1].
\]

Therefore, this is not a Markov chain.

**Solution 8.2** Let us identify the set \( a, b, c \) with \( 1, 2, 3 \). Then, from the diagram we can get the following transition matrix

\[
P = \begin{pmatrix}
0 & 1 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix}
\]

We know that \( P[X_n = a] = p_{n,a}^{(n)} = P^n(1,1) \). Then we need to calculate \( P^n \). We see that this matrix is diagonalizable since it has different eigenvalues. Indeed, it characteristic equation is given by

\[
0 = \det(\lambda I - P) = \lambda \left(\lambda - \frac{1}{2}\right)^2 - \frac{1}{4} = \frac{1}{4} (\lambda - 1)(4\lambda^2 + 1)
\]

and its eigenvalues are \( 1, i/2, -i/2 \). Hence, there exists an invertible matrix \( U \) such that

\[
P = U \begin{pmatrix}
1 & 0 & 0 \\
0 & i/2 & 0 \\
0 & 0 & -i/2
\end{pmatrix} U^{-1}
\]

and then

\[
P^n = U \begin{pmatrix}
1 & 0 & 0 \\
0 & (i/2)^n & 0 \\
0 & 0 & (-i/2)^n
\end{pmatrix} U^{-1}
\]
This implies that $P^n(1, 1) = x + y(i/2)^n + z(-i/2)^n$ for some constants $x, y, z$. We can calculate the value of these constants by using the first steps of our chain

$$1 = P^0(1, 1) = x + y + z$$
$$0 = P^1(1, 1) = x + iy/2 - iz/2$$
$$0 = P^2(1, 1) = x - y/4 - z/4.$$  

This give us $x = 1/5$, $y = (i - 2)/5$ and $z = (2 - i)/5$. Therefore

$$P^n(1, 1) = \frac{1}{5} + \frac{i - 2}{5} \left(\frac{1}{2}\right)^n \left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right) + \frac{2 - i}{5} \left(\frac{1}{2}\right)^n \left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)$$

$$= \frac{1}{5} + \left(\frac{1}{2}\right)^n \left(\frac{4}{5}\cos\frac{n\pi}{2} - \frac{2}{5}\sin\frac{n\pi}{2}\right).$$

**Solution 8.3**

(a) This is a Markov chain since the probability of adding a new value at time $n + 1$ depends on the number of values we have seen up to time $n$.

$$p_{i,j} = \begin{cases} \frac{N+1-j}{N} & \text{if } j = i + 1, \\ \frac{i}{N} & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Since $\xi_1, \xi_2, \ldots$ are i.i.d. uniform random variables on $\{1, \ldots, N\}$, we have that

$$P[X_n = i] = P[\{\xi_1, \ldots, \xi_n\} = i] = \sum_{J \subset \{1, \ldots, N\}} P[\{\xi_1, \ldots, \xi_n\} = J, |J| = i] = \binom{N}{i} P[\{\xi_1, \ldots, \xi_n\} = \{1, \ldots, i\}].$$

Let us call $\mathcal{X}_n = \{\xi_1, \ldots, \xi_n\}$ and $I = \{1, \ldots, i\}$, then

$$P[\mathcal{X}_n = I] = P[\mathcal{X}_n \subset I] - P[\mathcal{X}_n \subset I, \bigcup_{k=1}^i \{k \notin \{\xi_1, \ldots, \xi_n\}\}]$$

We know that $P[1, \ldots, k \notin \mathcal{X}_n, \mathcal{X}_n \subset I] = P[\xi_1, \ldots, \xi_n \in \{k+1, \ldots, i\}] = \left(\frac{i-k}{N}\right)^n$. Since there are $\binom{i}{k}$ ways of choosing the elements that do not appear in $\mathcal{X}_n$ and using the Inclusion-Exclusion principle, we have that

$$P[\mathcal{X}_n \subset I, \bigcup_{k=1}^i \{k \notin \{\xi_1, \ldots, \xi_n\}\}] = \sum_{k=1}^n (-1)^{k-1} \binom{i}{k} \left(\frac{i-k}{N}\right)^n.$$ 

Since $P[\mathcal{X}_n \subset I] = \left(\frac{i}{N}\right)^n$, we can put everything together to get

$$P[X_n = i] = \binom{N}{i} \sum_{k=0}^n (-1)^k \binom{i}{k} \left(\frac{i-k}{N}\right)^n.$$