# The modular group and the fundamental domain

Seminar on Modular Forms Spring 2019

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## 1 The Group $\mathrm{SL}_2(\mathbb{Z})$ and the fundamental domain

**Definition 1.** For a commutative Ring $R$ we define $\mathrm{GL}_2(R)$ as the following set:

$$\mathrm{GL}_2(R) := \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ for which } \det(A) = ad - bc \in R^* \right\}. \tag{1}$$

We define $\mathrm{SL}_2(R)$ to be the set of all $B \in \mathrm{GL}_2(R)$ for which $\det(B) = 1$.

**Lemma 1.** *$\mathrm{SL}_2(R)$ is a subgroup of $\mathrm{GL}_2(R)$.*

*Proof.* Recall that the kernel of a group homomorphism is a subgroup. Observe that det is a group homomorphism $\det : \mathrm{GL}_2(R) \to R^*$ and thus $\mathrm{SL}_2(R)$ is its kernel by definition. □

Let $R = \mathbb{R}$. Then we can define an action of $\mathrm{SL}_2(\mathbb{R})$ on $\bar{\mathbb{C}}$ ( $= \mathbb{C} \cup \{\infty\}$ ) by

$$A.z := \frac{az+b}{cz+d} \text{ and } A.\infty := \frac{a}{c}, \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(R),\ z \in \mathbb{C}. \tag{2}$$

**Definition 2.** The upper half-plane of $\mathbb{C}$ is given by $\mathbb{H} := \{z \in \mathbb{C} \mid Im(z) > 0\}$.

Restricting this action to $\mathbb{H}$ gives us another well defined action "." : $\mathrm{SL}_2(\mathbb{R}) \times \mathbb{H} \mapsto \mathbb{H}$ called the fractional linear transformation. Indeed, for any $z \in \mathbb{H}$ the imaginary part of $A.z$ is positive:

$$Im(A.z) = Im\Big(\frac{az+b}{cz+d}\Big) = Im\Big(\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}\Big) = \frac{Im(z)}{|cz+d|^2} > 0. \tag{3}$$

**Lemma 2.** *For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ the map $\mu_A : \mathbb{H} \to \mathbb{H}$ defined by $z \mapsto A.z$ is the identity if and only if $A = \pm I$.*

*Proof.* Assume $z = \frac{az+b}{cz+d}$ for all $z \in \mathbb{H}$ which reduces to $cz^2 + (d-a)z + b = 0$. Having assumed $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is in $\mathrm{SL}_2(\mathbb{R})$, it follows that the only solutions satisfying $ad - bc = 1$ are $b = c = 0$ and $a = d$ so $a = d = \pm 1$ and hence $\pm I$ are the only elements of $\mathrm{SL}_2(\mathbb{R})$ that act trivially on the upper half plane. □

**Definition 3.** The full modular group is given by:

$$\Gamma := \mathrm{SL}_2(\mathbb{Z}) \leq \mathrm{SL}_2(\mathbb{R}). \tag{4}$$

In order to make the group act faithfully on $\mathbb{H}$ (recall that this means that there are no group elements $A \in \Gamma \backslash \{Id\}$ such that $A.z = z, \quad \forall z \in \mathbb{H}$). We define the modular group $\bar{\Gamma} := \Gamma/\{\pm I\}$.

The fact that $\Gamma$ is indeed a subgroup of $\mathrm{SL}_2(\mathbb{R})$ can be verified through checking that the inverse of an element in $\Gamma$ lies in $\Gamma$ too: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} = \frac{1}{ad-bc}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ and $\frac{1}{ad-bc} = 1 \in \mathbb{Z}$ by assumption. Also since $\Gamma$ is a group it follows that $\bar{\Gamma} = \Gamma/\{\pm\, I\}$ is a group, as $\{\pm I\}$ is normal in $\Gamma$.

**Definition 4.** Let $N > 0$. Then we define the principle subgroup of level $N$ by

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | a \equiv d \equiv 1 (\mathrm{mod}\ N), b \equiv c \equiv 0 (\mathrm{mod}\ N) \right\}. \tag{5}$$

First we show some basic properties of these subgroups:

**Lemma 3.** *Let $N > 0$. Then $\Gamma(N)$ is a normal subgroup of $\Gamma$.*

*Proof.* We claim that $\Gamma(N)$ is the kernel of a group homomorhism $\psi$ and hence normal. As a candidate we choose $\psi$ as follows:

$$\psi : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \tag{6}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a\, (\mathrm{mod}\ N) & b\, (\mathrm{mod}\ N) \\ c\, (\mathrm{mod}\ N) & d\, (\mathrm{mod}\ N) \end{pmatrix}. \tag{7}$$

This is indeed a group homomorphism, as it is one in every component. Its kernel is then exactly $\Gamma(N)$. We are done. $\square$

<u>**Remark:** For all</u> $N > 2$, the kernel $\bar{\Gamma}(N)$ of the homomorphism $\psi : \overline{\mathrm{SL}_2(\mathbb{Z})} \to \overline{\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})}$ defined as in Lemma 3 is in fact equal to $\Gamma(N)$. Indeed, for $N > 2$ we have $-1 \not\equiv 1\, mod(N)$ and thus $-I \notin \Gamma(N)$.

**Definition 5.** Let $C$ subgroup of $\Gamma$. We call $C$ a congruence subgroup of level $N$ if $C$ contains $\Gamma(N)$.

Similarly we may call a subgroup $C'$ of $\bar{\Gamma}$ a congruence subgroup of level $N$ if it contains $\bar{\Gamma}(N)$.

The most important examples of congruence subgroups, $\Gamma_0(N)$ and $\Gamma_1(N)$ will be defined here but are of little relevance for this lecture:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma | c \equiv 0\, (\mathrm{mod}\ N) \right\} \tag{8}$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) | a \equiv 1\, (\mathrm{mod}\ N) \right\}. \tag{9}$$

The fact that these are indeed subgroups with $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N) \subset \Gamma$ will not be proven here. We now continue with the most important definition of the lecture, the fundamental domain.

**Definition 6.** Let $G$ be a group acting on a topological space $X$. A fundamental domain $F$ for the action of $G$ is a (closed) subset, such that exactly one point of each orbit is contained in (the interior of) $F$.

Applying this to our situation we call a closed subset $F \subset \mathbb{H}$ a fundamental domain for a subgroup $G$ of $\Gamma$ if every $z \in \mathbb{H} \backslash F$ is $G$-equivalent to one point in the interior of $F$ (denoted $F^\circ$) and no two points $z_1, z_2 \in F^\circ$ are $G$-equivalent.

We will prove and describe the fundamental domain of the whole group $\Gamma$ itself before turning to subgroups.

**Theorem 1.** *A fundamental domain of* $\Gamma$ *under fractional linear transformations (2) is given by*

$$F := \{z \in \mathbb{H} | -\frac{1}{2} \leq Re(z) \leq \frac{1}{2} \text{ and } |z| \geq 1\}.$$

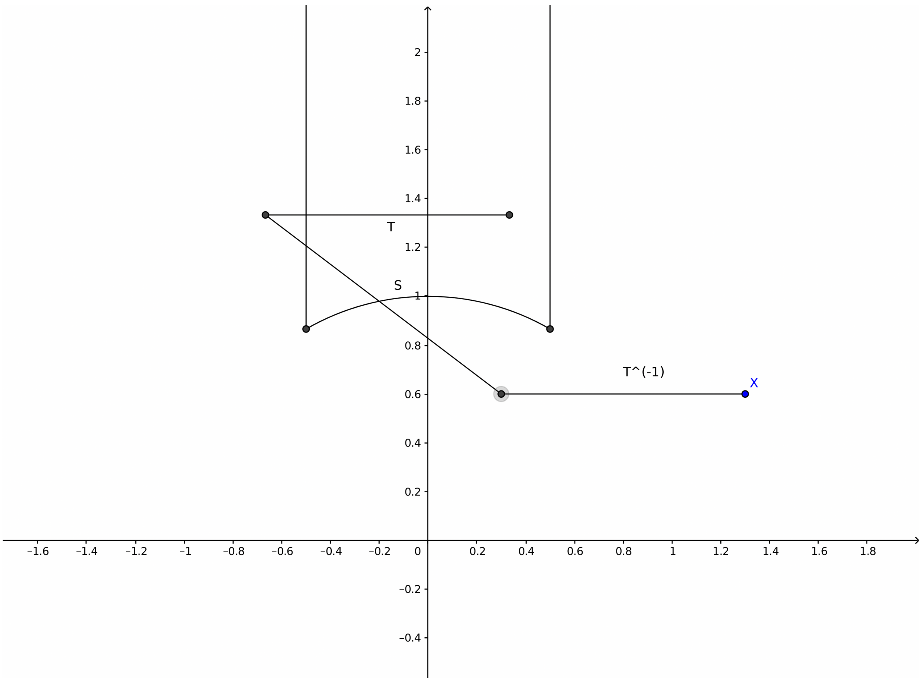


Figure 1: Fundamental domain of $\Gamma$ and approximating $\gamma = T \circ S \circ T^{-1}$ such that $\gamma.x \in F$.

*Proof.* The idea is to fix any point $z \in \mathbb{H}$ and then to find a combination of translations and inversions until $z$ is mapped to $F$. This process is demonstrated for some $x$ in Figure 1.

Fix $z \in \mathbb{H}$. Define $\Gamma'$ to be the subgroup generated by $T$ and $S$, where $T$ and $S$ are:

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ so that } T.z = z + 1, \tag{10}$$

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ so that } S.z = -\frac{1}{z}. \tag{11}$$

Now let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma'$. Then by (3) we have $Im(\gamma.z) = Im(z)|cz+d|^{-2}$ and thus $Im(\gamma.z) > 0$. By choosing the factors $c$ and $d$ so that $|cz+d|$ is as small as possible but not zero. This is possible as $cz+d$ must be on the lattice generated by 1 and $z$. Hence $|cz+d|$ must be bigger or equal than one of the following: $\{|\pm z+0|, |\pm z \pm 1|, |0+\pm 1|\}$ by construction. But this set is finite hence has a minimum. So we are able to find a $\gamma$ such that $Im(\gamma.z)$ is maximal. In a next step find a suitable $j \in \mathbb{N} \cup \{0\}$ so that $(T^j\gamma).z \in \{z' \in \mathbb{H}| -\frac{1}{2} \leq Re(z) \leq \frac{1}{2}\}$. Define $T^j\gamma =: \gamma'$ and claim that $\gamma'.z$ lies in $F$.

If this was not the case, so $|\gamma'.z| < 1$, then again by (3) we would have $Im((S\gamma').z) = Im(\gamma'.z)|\gamma'.z|^{-2} > Im(\gamma'.z)$ contradicting the maximality requirement we set above. Hence $\exists \gamma \in \Gamma'$ such that $\gamma.z \in F, \quad \forall z \in \mathbb{H}$.

We now have to prove that no two points in the interior of $F$ share the same Orbit. For this assume $z_1, z_2 \in F$ and wlog that $Im(z_2) \geq Im(z_1)$ and lastly that there exists an $A \in \Gamma$ so that $z_2 = A.z_1$.

Having assumed that it follows that

$$Im(z_2) = Im(A.z_1) = Im(z_1)|cz_1 + d|^{-2} \geq Im(z_1).$$

by (3) and hence $|cz_1 + d|^2 \leq 1$. As all entries in $A$ must lie in $\mathbb{Z}$ and $Im(z_1) \geq \frac{\sqrt{3}}{2}$, it follows that the absolute value of $c$ must be less than 2 otherwise $1 \leq Im(cz_1) = Im(cz_1 + d) \leq |cz_1 + d|$. Moreover, we have $|d| - \frac{1}{2} \leq |d + \mathrm{Re}(z)c| \leq |cz + d| \leq 1$. This slims down the possibilities for $A$ and we can deal with them case by case.

Case 1: $c = 0$, $d = \pm 1$. By enforcing the $ad - bc = 1$ constraint we see that then $A$ or $-A$ must be equal to $T^j = \left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right)$. But for $A.z_1$ to lie in $F$ again only $j \in \{-1, 0, 1\}$ come in question. For $j = 0$, we have $z_1 = z_2$. For $j = \pm 1$ recall the definition of $T$ and we see that $z_2$ and $z_1$ must lie on the boundary lines $Re(z) = \pm\frac{1}{2}$ of $F$ and hence not in the interior of $F$.

Case 2: $c = \pm 1$, $d = 0$. Then $z_1$ lies on the unit circle, as we have $|cz_1| \leq 1$ and $|z_1| \geq 1$. To satisfy $ad-bc = 1$, $A$ has to be of the form $A = \pm\left(\begin{smallmatrix} a & -1 \\ 1 & 0 \end{smallmatrix}\right) = T^aS$ for some $a \in \mathbb{Z}$. For $|a| > 2$ we see that $T^aS.z_1 \notin F$, thus $a \in \{-1, 0, 1\}$.

For $a = 0$ we get $A = \pm S$ and hence that $z_1, z_2$ lie in $\mathbb{S}^1 \cap F$ and are symmetric with respect to $i\mathbb{R}$ as $|z_1| = 1$.

For $a = 1$ we have $A = \pm TS$, but then $z_1 = \frac{1}{2} + \frac{\sqrt{-3}}{2}$. The case $a = -1$ is similar: We get $A = \pm ST$ and $z_1 = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$.

Case 3: Suppose $c = d = \pm 1$. Since $|cz+d| \leq 1$, we see that $z_1 = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$, as for any other $z_1 \in F$ we find that $|\pm z_1 \pm 1| > 1$.

By the condition $\det(A) = 1$ it follows that: $a - b = 1 \implies b = -1 + a$. And hence of the form $A = \left(\begin{smallmatrix} 1 & -1+a \\ 1 & 1 \end{smallmatrix}\right) = T^a \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$.

By assumption we want $A.z_1 \in F$. One solution is $a = 0$ but then $z_1 = z_2 = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$. For the case $a = 1$ we see that $z_2 = z_1 + 1 = \frac{1}{2} + \frac{\sqrt{-3}}{2}$, so all viable points lie on the boundary again as for any other choice of $a$ one can think of $T^a$ as the map that moves $z_1$ too far away (or by computing it).

Case 4: $c = -d = \pm 1$ and $z_1 = \frac{1}{2} + \frac{\sqrt{-3}}{2}$ Here we proceed in the same way as in case 3 and see the same result - viable pairs $z_1, z_2$ must lie in the boundary of $F$.

We thus conclude: In no case do $z_1$ and $z_2$ belong to the interior unless $A = \pm I$ and $z_1 = z_2$. This concludes the proof. □

**Corollary 1.** *For any $z \in F$ we can give an explicit definition of the Stabilizer of $z$ under $\Gamma$, denoted $\Gamma_z$:*

- $\Gamma_z = \{I, S\}$ *for* $z = i$.
- $\Gamma_z = \{I, ST, (ST)^2\}$ *for* $z = \omega := -\frac{1}{2} + \frac{\sqrt{-3}}{2}$.
- $\Gamma_z = \{I, TS, (TS)^2\}$ *for* $z = -\bar{\omega}$.
- $\Gamma_z = \{\pm I\}$ *for all other* $z \in F$.

**Corollary 2.** *The group $\bar{\Gamma}$ is generated by the two elements $S$ and $T$. Hence any fractional linear transformation can be written as a word in letters $S$ and $T$.*

*Proof.* Let again $\Gamma'$ be the subgroup of $\Gamma$ spanned by $T$ and $S$. Let $z$ be in the interior of $F$ and $A \in \Gamma$. Consider $A.z \in \mathbb{H}$. By the first part of the proof of the theorem above we can find a $A' \in \Gamma'$ so that $(A'A).z \in F$. However as $z$ lies in the interior of $F$ its Stabilizer is given by $\pm I$ it follows that $A'A = \pm I \iff A' = \pm A^{-1}$. Hence, any $A \in \Gamma$ lies in $\Gamma'$ up to a sign, proving the corollary. □

To get a one to one correspondence between the orbits and some modified set of $F$ we can identify $\Gamma$-equivalent points on the boundary of $F$ with each other. This can be visualized as identifying $\frac{1}{2} + iy$ with $-\frac{1}{2} + iy$ for any $y \geq \frac{\sqrt{3}}{2}$ and the points on the "round" part of the boundary with each other, formally $e^{2\pi i\vartheta}$ with $e^{2\pi i(\frac{1}{2}-\vartheta)}$ for all $\vartheta$ between one sixth and one third. This updated $F$ gives us a one to one correspondence with the the set of Orbits of $\Gamma$ acting on $\mathbb{H}$ and we denote it by $\Gamma\backslash\mathbb{H}$.

**Definition 7.** The points $\mathbb{Q} \cup \{\infty\}$ in $\bar{\mathbb{H}}$ are called cusps.

**Lemma 4.** $\Gamma$ *permutes the cusps transitively.*

*Proof.* If we show that every rational number $\frac{a}{c}$ lies in the orbit of $\infty$ we are done. So assume $\frac{a}{c} \in \mathbb{Q}$ with $a, c \in \mathbb{Z}$ then we can solve $ad - bc = 1$ for $d$ and $b$. This gives us a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and by the definition of the fractional linear transformation (2) we have $A.\infty = \frac{a}{c}$ as desired. □

We now turn to fundamental domains of subgroups $\Gamma' \subset \Gamma$ of finite index ($[\Gamma : \Gamma'] = n < \infty$). Hence $\Gamma = \bigsqcup_{i=1}^{n} \alpha_1 \Gamma'$ with $\alpha_i \in \Gamma$.

**Lemma 5.** *A fundamental domain $F'$ of a subgroup $\Gamma'$ of $\Gamma$ is given by $F' = \bigcup_{i=1}^{n} \alpha_i^{-1}.F$.*

*Proof.* Similar to the proof of Theorem 1 we first show that every $z \in \mathbb{H}$ is $\Gamma'$-equivalent to a point in $F'$. Therefore, let $z \in \mathbb{H}$. By Theorem 1 we can find some $\gamma \in \Gamma$ such that $\gamma.z \in F$. Hence for some $i$, we have $\gamma = \alpha_i \gamma'$ with $\gamma' \in \Gamma'$ this implies $\gamma'.z = \alpha_i^{-1}\gamma.z \in \alpha_i^{-1}.F \subset F'$. Again we have to show that no two elements in the interior of $F'$ are $\Gamma'$-equivalent. Assume that $z_1, z_2 \in F'$ and there exists a $\gamma \in \Gamma'$ such that $z_2 = \gamma.z_1$ also choose $i, j$ s.t $z_1 \in \alpha_i^{-1}.F$ and $z_2 \in \alpha_j^{-1}.F$. Define $f_1 := \alpha_i.z_1$ and $f_2 := \alpha_j.z_2$ but this would also mean that we have found two $\Gamma$-equivalent points in $F$ by $f_2 = \alpha_j \gamma \alpha_i^{-1}.f_1$ contradicting Theorem 1, as $f_1, f_2 \in F$. $\square$

**Exercise:** Compute a fundamental domain of $\Gamma(2)$.

# 2 Different Realizations of $\mathbb{H}$

## 2.1 $\mathbb{H}$ as a quotient of $\mathrm{SL}_2$

In this section, we wish to identify $\mathbb{H}$ with the quotient $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$. We will introduce an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ and then look at a special Orbit - namely the one of $i$. Afterwards we quotient out and define another group action. Theorem 2 then links the two actions and thus completes the statement.

**Lemma 6.** *The group action of* $\mathrm{SL}_2(\mathbb{R})$ *on* $\mathbb{H}$ *defined in equation (2) is transitive.*

*Proof.* In order to show it acts transitively we fix any $z = x + iy \in \mathbb{H}$ and wish to find an element $A \in \mathrm{SL}_2(\mathbb{R})$ so that $A_z.i = z$. Define $A_z$ by

$$A_z := \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \sqrt{y}^{-1} \end{pmatrix}. \tag{12}$$

$A_z$ lies in the group $\mathrm{SL}_2(\mathbb{R})$ as it has determinant 1. By computing we see that $A_z.i = (\sqrt{y}i + \frac{x}{\sqrt{y}})\sqrt{y} = x + iy = z$. Hence, we can conclude that such an element $A_z$ exists for all $z \in \mathbb{H}$ so the group action is transitive. $\square$

**Lemma 7.**

$$Stab(i) = \mathrm{SO}_2(\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) | A^T A = I \text{ , } \det(A) = 1 \right\}.$$

*Proof.* Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ so that $A.i = i$ Then we see by (2) that $\frac{ai+b}{ci+d} = i$, equivalently $ai+b = di+c$ and because all entries of the matrix are real it follows that $a = d$ and $b = -c$ which is a necessary and sufficient requirement for $A$ to lie in $\mathrm{SO}_2(\mathbb{R})$. $\square$

For $\mathbb{R}$, $\mathrm{SO}_2(\mathbb{R})$ is not a normal subgroup of $\mathrm{SL}_2(\mathbb{R})$. In order to see this, we recall that a if a subgroup $H$ of a group $G$ is normal, then for any $g \in G$ and $h \in H$, the product $ghg^{-1}$ lies in $H$. In our case this reduces to saying that for $A \in \mathrm{SL}_2(\mathbb{R})$ and $M \in \mathrm{SO}_2(\mathbb{R})$, the subgroup would be normal if $(AMA^{-1})(AMA^{-1})^T = I$. Finding an example where this does not hold is easy. Thus the quotient $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ is only a set without induced group structure. However, a group can act on this set.

**Lemma 8.** *The group* $\mathrm{SL}_2(\mathbb{R})$ *acts through* $\mu$ *on* $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$*:*

$$\mu : \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \tag{13}$$
$$(A, M\mathrm{SO}_2(\mathbb{R})) \longmapsto \mu_A(M\mathrm{SO}_2(\mathbb{R})) = AM\mathrm{SO}_2(\mathbb{R}).$$

*This action is transitive.*

*Proof.* We first show that the group action is well defined: Let $M\mathrm{SO}_2(\mathbb{R}) = N\mathrm{SO}_2(\mathbb{R}) \in \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ be two cosets defined by different elements. Then $M = NV$ for $V \in \mathrm{SO}_2(\mathbb{R})$.Thus:

$$\mu_A(M\mathrm{SO}_2(\mathbb{R})) = AM\mathrm{SO}_2(\mathbb{R}) = ANV\mathrm{SO}_2(\mathbb{R}) \tag{14}$$
$$= AN\mathrm{SO}_2(\mathbb{R}) = \mu_A(N\mathrm{SO}_2(\mathbb{R}))$$

To show that we indeed have a group action observe that $I \in \mathrm{SO}_2(\mathbb{R})$ and thus $\mu_A \circ \mu_I = \mu_I \circ \mu_A$ and $\mu_A \circ \mu_B = \mu_{AB}$ which establishes the group action. TO show transitiveness, we show that for all $M\mathrm{SO}_2(\mathbb{R}) \in \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ there exists a $A$ in $\mathrm{SL}_2(\mathbb{R})$ so that $\mu_A(M\mathrm{SO}_2(\mathbb{R})) = \mathrm{SO}_2(\mathbb{R})$. Define $A = M^{-1}$ and compute:

$$\mu_{M^{-1}}(M\mathrm{SO}_2(\mathbb{R}) = M^{-1}M\mathrm{SO}_2(\mathbb{R}) = \mathrm{SO}_2(\mathbb{R}). \tag{15}$$

□

**Theorem 2.** *There is a bijection* $\psi$ *between* $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ *and* $\mathbb{H}$*. The map* $\psi$ *is defined as follows:*

$$\psi : \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \to \mathbb{H} \tag{16}$$
$$M\mathrm{SO}_2(\mathbb{R}) \longmapsto M.i\ .$$

*In particular, the following diagram commutes for all* $A \in \mathrm{SL}_2(\mathbb{R})$*.*

$$\begin{tikzcd}
\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \arrow[r, "\psi"] \arrow[d, "\mu_A"] & \mathbb{H} \arrow[d, "A."] \\
\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \arrow[r, "\psi"] & \mathbb{H}
\end{tikzcd}$$

*Proof.* We will start by recalling the orbit stabilizer theorem. Assume $G$ is a group acting on a set $X$. Then the orbit stabilizer theorem tells us that there is a bijection between the orbit of $x \in X$ and $G/G_x$ where the orbit of $x$, denoted $\mathrm{orb}(x)$ is defined as $\{g.x \mid g \in G\}$ and $G_x := \{g \in G \mid g.x = x\}$. In fact, the standard proof of the theorem defines the map $\phi : G/G_x \to \mathrm{orb}(x)$, $hG_x \mapsto h.x$ and then shows it is a bijection.

Now we apply this theorem to our situation: $G = \mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by linear fractional transformations and we have already seen in Lemma 7 that $G_i = \mathrm{SO}_2(\mathbb{R})$ and $\mathrm{orb}(i) = \mathbb{H}$. This shows that there is indeed a bijection, which is given by $\psi : \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \to \mathbb{H}$, $M\mathrm{SO}_2(\mathbb{R}) \mapsto M.i$ □

This realization is useful when trying to generalize modular forms, as it allows a generalization of the domain $\mathbb{H}$. See [3] for further reading on this topic.

## 2.2 $\mathbb{H}$ as the group $\mathrm{SP}_2(\mathbb{R})$

In this section, we want to study another identification of the upper half plane, this time through symmetric positive definite matrices.

**Definition 8.** Any $n \times n$ matrix is called symmetric positive definite (spd) if it is symmetric : $A^T = A$, and positive definite: for all $v \in \mathbb{R}^n \backslash \{0\}$ the scalar $v^T A v$ is strictly positive. We write $A > 0$.

**Definition 9.** The subgroup $SP_2(\mathbb{R})$ of $\mathrm{GL}_2(\mathbb{R})$ is defined by

$$SP_2(\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}) \text{ for which } A > 0 \text{ and } \det(A) = 1 \right\}. \tag{17}$$

Here is a criterion that will make it a lot easier to construct matrices in $SP_2(\mathbb{R})$.

**Lemma 9.** $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in SP_2(\mathbb{R}) \iff a > 0$ *and* $\det(A) = 1$.

We define two maps, then show that they are well-defined and each others inverse.

**Definition 10.** We define $F : \mathbb{H} \to SP_2(\mathbb{R})$ by $z = x + iy \mapsto F_z = \frac{1}{y}\left(\begin{smallmatrix} 1 & -x \\ -x & x^2+y^2 \end{smallmatrix}\right)$. Additionally we define $\omega : SP_2(\mathbb{R}) \to \mathbb{H}$ by $M = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \mapsto \omega(M) = \frac{1}{a}(-b+i) \in \mathbb{H}$.

**Lemma 10.** *The two functions $F$ and $\omega$ are both well-defined and each others inverse, making both maps bijective.*

*Proof.* First, $F$ is well defined by the criterion we stated in Lemma 9: $\det(F_z) = 1$ and $\frac{1}{y} > 0$. The map $\omega$ is well defined too; again due to our criterion, $a > 0$. Therefore $\omega(S)$ for $S \in SP_2(\mathbb{R})$ lies in the upper half plane because the term $\frac{1}{a}i$ is strictly positive.

By computing the term $\omega(F_z)$ for $z = x + iy$ we obtain $z$ again:

$$\omega(F_z) = \omega\Big(\frac{1}{y}\begin{pmatrix} 1 & -x \\ -x & x^2+y^2 \end{pmatrix}\Big) = y\Big(\frac{x}{y} + i\Big) = z. \tag{18}$$

Next, if we let $M = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \in SP_2(\mathbb{R})$ then :

$$F_{\omega(M)} = F_{\frac{1}{a}(-b+i)} = a\begin{pmatrix} 1 & \frac{b}{a} \\ \frac{b}{a} & \frac{b^2+1}{a^2} \end{pmatrix} = \begin{pmatrix} a & b \\ b & \frac{b^2+1}{a} \end{pmatrix}. \tag{19}$$

The term $\frac{b^2+1}{a}$ is exactly the value we get for $c$ if we fix the entries $a$ and $b$ of the matrix and enforce $\det(M) = 1$ through $ac - b^2 = 1$.

This proves that the maps are inverses of each other. Thus they are both bijective. $\square$

We have already elaborated how $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ through the fractional linear transformations and established that $SP_2(\mathbb{R})$ has a bijective correspondence to $\mathbb{H}$. Thus, we can define an action on $SP_2(\mathbb{R})$ by going to $\mathbb{H}$ and back again:

$$\alpha : \mathrm{SL}_2(\mathbb{R}) \times SP_2(\mathbb{R}) \to SP_2(\mathbb{R}) \tag{20}$$

$$(A, M) \longmapsto F_{A.(\omega(M))} \tag{21}$$

However, this definition is tedious and we wish for a more explicit description.

**Lemma 11.** *Let $M = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right) \in SP_2(\mathbb{R})$ and the map $\omega$ as above. Then the number $\omega(M) \in \mathbb{H}$ solves the equivalent equations*

$$a\omega^2 + 2b\omega + c = 0 \iff \begin{pmatrix} \omega \\ 1 \end{pmatrix}^T M \begin{pmatrix} \omega \\ 1 \end{pmatrix} = 0. \tag{22}$$

*Proof.* We use the standard quadratic formula :

$$\frac{-2b \pm \sqrt{4b^2 - 4ac}}{2a} = -\frac{b}{a} \pm \frac{\sqrt{b^2 - ac}}{a} =^{(*)} \frac{-b \pm i}{a} = \omega(M). \tag{23}$$

(*) is due to the condition $\det(M) = 1$.

Assume now that $z \in \mathbb{H}$ solves the equation (22). Then we claim that $z = \frac{1}{a}(-b + i)$. Indeed:

$$\begin{aligned} 0 &= az^2 + 2bz + c && (24)\\ &= az^2 + 2bz + \frac{1 + b^2}{a} && (25)\\ &= a^2 z^2 + 2abz + 1 + b^2 && (26)\\ &= (az)^2 + 2b(az) + 1 + b^2 && (27)\\ &= v^2 + 2bv + 1 + b^2 && (28) \end{aligned}$$

We solve the equation for $v$ and see that $v = (-b \pm i)$. Focusing on the $v$ that lies in in $\mathbb{H}$, we see that

$$\begin{aligned} v = az &= -b + i && (29)\\ z &= \frac{1}{a}(-b + i) && (30) \end{aligned}$$

which shows our claim. □

**Lemma 12.** *The group action $\beta$ of $\mathrm{SL}_2(\mathbb{R})$ on $SP_2(\mathbb{R})$ defined by*

$$\beta(A, M) := A * M = (A^{-1})^T M A^{-1}, \tag{31}$$

*for $A \in \mathrm{SL}_2(\mathbb{R})$ and $M \in SP_2(\mathbb{R})$ is the same group action as $\alpha$ and thus in particular transitive.*

*Proof.* We claim that showing that the group actions are the same is equivalent to showing $A.\omega(M) = \omega(A * M)$. Indeed, using Lemma 10 for the second equivalence we see

$$\begin{aligned} A.\omega(M) &= \omega(A * M)\\ &\iff\\ \omega^{-1}(A.\omega(M)) &= A * M\\ &\iff\\ F_{A.\omega(M))} &= A * M\\ &\iff\\ \alpha(A, M) &= \beta(A, M). \end{aligned}$$

To prove $A.\omega(M) = \omega(A * M)$, we first define:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } A * M = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \tag{32}$$

$$\theta := \omega(A * M) := \omega\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} * M \right) := \omega\left( \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \right) \text{ and } v := A^{-1}.\theta. \tag{33}$$

By computing it follows that $v = \frac{d\theta - b}{-c\theta + a}$, using the inverse and the fractional linear transformation definition.

By Lemma 11, $\theta$ solves the equation $\left(\begin{smallmatrix} \theta \\ 1 \end{smallmatrix}\right)^T (A^{-1})^T M A^{-1} \left(\begin{smallmatrix} \theta \\ 1 \end{smallmatrix}\right) = 0$. We claim that it also solves

$$\begin{pmatrix} v \\ 1 \end{pmatrix}^T M \begin{pmatrix} v \\ 1 \end{pmatrix} (c\theta + a)^2 = 0. \tag{34}$$

Indeed, $A^{-1}\left(\begin{smallmatrix} \theta \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} d\theta - b \\ -c\theta + a \end{smallmatrix}\right) = (-c\theta + a)\left(\begin{smallmatrix} \frac{d\theta - b}{-c\theta + a} \\ 1 \end{smallmatrix}\right) = (-c\theta + a)\left(\begin{smallmatrix} v \\ 1 \end{smallmatrix}\right)$ and thus the claim proves correct by plugging in the term we obtained. We are allowed to divide both sides by $(c\theta + a)^2 = (-c\theta + a)(c\theta + a)$ as $(-c\theta + a)$ never vanishes. Thus, this equation holds true:

$$\begin{pmatrix} v \\ 1 \end{pmatrix}^T M \begin{pmatrix} v \\ 1 \end{pmatrix} = 0. \tag{35}$$

Lemma 11 tells us that $v = \omega(M)$ which then due to our definition $v = A^{-1}.\theta$ implies:

$$A.(\omega(M)) = \theta = \omega(A * M) \tag{36}$$

showing that $A.\omega(M) = \omega(A * M)$, and thus ending the proof. $\square$

This realization of the upper half plane will be useful when studying Siegel modular forms, as they allow another type of generalization. For further reading on this topic see for example [4].

# References


[1] Jan Hendrik Bruinier, Gerard van der Geer, Günter Harder, and Don Zagier. *The 1-2-3 of modular forms*. Universitext. Springer-Verlag, Berlin, 2008. Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004, Edited by Kristian Ranestad.

[2] Fred Diamond and Jerry Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[3] Stephen Gelbart. An elementary introduction to the Langlands program. *Bull. Amer. Math. Soc. (N.S.)*, 10(2):177–219, 1984.

[4] Helmut Klingen. *Introductory lectures on Siegel modular forms*, volume 20 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[5] Neal Koblitz. *Introduction to elliptic curves and modular forms*, volume 97 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.