MODULAR FORMS : DEFINITIONS AND EXAMPLES

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1. Definitions of Modular Forms

We dedicate a first section to the rigorous definition of a modular form. We also prove some useful criteria and basic properties. Recall that $SL(2,\mathbb{Z})$ acts on the upper half plane $\mathbb{H}$ via Möbius transformations. A first step towards the definition of a modular form is to turn this action into an action of $SL(2,\mathbb{Z})$ on the space of complex-valued function on $\mathbb{H}$. To this end, we introduce the notation j.

Definition 1.1. For every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ and $\tau \in \mathbb{H}$, we note $j(\gamma,\tau) = c\tau + d$.

Lemma 1.2. *For every γ_1, $\gamma_2 \in SL(2,\mathbb{Z})$, $\tau \in \mathbb{H}$, $j(\gamma_1\gamma_2,\tau) = j(\gamma_1,\gamma_2\tau)j(\gamma_2,\tau)$.*

Proof. A first observation is the following : for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ and $\tau \in \mathbb{H}$,

$$\gamma \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix} = (c\tau + d) \begin{pmatrix} \frac{a\tau+b}{c\tau+d} \\ 1 \end{pmatrix} = j(\gamma,\tau) \begin{pmatrix} \gamma\tau \\ 1 \end{pmatrix}.$$

Now, we can compute on the one side :

$$\gamma_1\gamma_2 \begin{pmatrix} \tau \\ 1 \end{pmatrix} = j(\gamma_1\gamma_2,\tau) \begin{pmatrix} \gamma_1\gamma_2 \\ 1 \end{pmatrix}.$$

On the other side,

$$\gamma_1\gamma_2 \begin{pmatrix} \tau \\ 1 \end{pmatrix} = j(\gamma_2,\tau)\gamma_1 \begin{pmatrix} \gamma_2\tau \\ 1 \end{pmatrix} = j(\gamma_2,\tau)j(\gamma_1,\gamma_2\tau) \begin{pmatrix} \gamma_1\gamma_2\tau \\ 1 \end{pmatrix}.$$

Therefore, we identify $j(\gamma_1\gamma_2,\tau) = j(\gamma_1,\gamma_2\tau)j(\gamma_2,\tau)$, as desired.

$\square$

We get an action of $SL(2,\mathbb{Z})$ on the complex-valued functions on $\mathbb{H}$ as follows.

Definition 1.3. For every $f : \mathbb{H} \to \mathbb{C}$, $\gamma \in SL(2,\mathbb{Z})$ and integer k, we note $f[\gamma]_k$ for the function

$$\begin{aligned} f[\gamma]_k : \mathbb{H} &\longrightarrow \mathbb{C}; \\ \tau &\longrightarrow j(\gamma,\tau)^{-k} f(\gamma\tau). \end{aligned}$$

Lemma 1.4. *Definition 1.3 specifies a right action of $SL(2,\mathbb{Z})$ on the complex-valued functions on $\mathbb{H}$.*

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Proof. We straightforwardly compute : for all $f : \mathbb{H} \to \mathbb{C}$, $\gamma \in SL(2,\mathbb{Z})$, $\tau \in \mathbb{H}$ and integer k,

$$\begin{aligned} f[\gamma_1\gamma_2]_k(\tau) &= j(\gamma_1\gamma_2,\tau)^{-k} f(\gamma_1\gamma_2\tau) \\ &= j(\gamma_2,\tau)^{-k} j(\gamma_1,\gamma_2\tau)^{-k} f(\gamma_1\gamma_2\tau) \\ &= j(\gamma_2,\tau)^{-k} f[\gamma_1]_k(\gamma_2\tau) \\ &= f[\gamma_1]_k[\gamma_2]_k(\tau). \end{aligned}$$

Since the identity matrix clearly acts trivially, we are done. ◻

Definition 1.5. Let $f : \mathbb{H} \to \mathbb{C}$ be a meromorphic function and k an integer. f is called *weakly modular of weight k* if

$$f[\gamma]_k(\tau) = f \text{ for all } \gamma \in SL(2,\mathbb{Z}). \tag{1}$$

We start with a criterion for weak modularity.

Proposition 1.6. *Let $f : \mathbb{H} \to \mathbb{C}$ be a meromorphic function. Then, f is weakly modular of weight k if and only if*

$$f(\tau+1) = f(\tau) \textit{ and } f(-1/\tau) = \tau^k f(\tau) \textit{ for all } \tau \in \mathbb{H}. \tag{2}$$

Proof. Suppose that first that f is weakly modular. Then, applying the relation 1 to the matrices

$$T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \qquad \text{and} \qquad S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

yields the relations (2).

We now suppose that f only satisfies the relations (2). Since the operators $[\gamma]_k$ define a right action of $SL(2,\mathbb{Z})$ on the complex-valued functions on $\mathbb{H}$, it is enough to show that $SL(2,\mathbb{Z})$ is generated by T and S. Observe first that

$$T^n = \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}.$$

Let now $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any element of $SL(2,\mathbb{Z})$ and $\Gamma = < S, T >$. Up to multiplying by $-Id = S^2 \in \Gamma$, we may assume that $d \geq 0$. We now split cases. If $c = 0$, the relation $det(\alpha) = 1$ forces $a = d = \pm 1$ and since $d \geq 0$, $\alpha = T^b \in \Gamma$. If $c \neq 0$, the Euclidean division gives $d = mc + n$, $0 \leq n < c$. Up to replacing m by $m\pm1$ if needed, we may assume that $|n| < \frac{c}{2}$. Thus, using

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -m \\ & 1 \end{pmatrix} = \begin{pmatrix} a & * \\ c & d - mc \end{pmatrix},$$

we see that we can multiply α by an element of Γ to get a matrix whose bottom row is (c, d') with $|d'| < \frac{c}{2}$. Now, multiplying by S switches the bottom elements up to a sign so we can induct this process to get $\beta \in \Gamma$ such that

$$\alpha\beta = \begin{pmatrix} a' & b' \\ & d' \end{pmatrix}.$$

We can now conclude as in the case $c = 0$. ◻

Example 1.7. As an exercise, one can apply this process to decompose the following matrix in a product of powers of T and S :

$$\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}.$$

Proposition 1.6 enables one to expand a weakly modular function as a Fourier series. The latter will enables us to define the notion of holomorphy at ∞. More precisely, we have :

Corollary 1.8. *Let $f : \mathbb{H} \to \mathbb{C}$ be holomorphic and weakly modular. Then,*

$$f(\tau) = \sum_{n \in \mathbb{Z}} f_n e^{2\pi i n \tau} \text{ for some } f_n \in \mathbb{C}.$$

Proof. Thanks to the relation (2), f is $\mathbb{Z}$-periodic and holomorphic. Thus the function $f_y : \mathbb{R} \to \mathbb{C};\ x \mapsto f(x+iy)$ is C^∞ and 1-periodic so we get a Fourier expansion

$$f_y(x) = \sum_{n \in \mathbb{Z}} a_n(y) e^{2\pi i n x}, \text{ where } a_n(y) = \int_0^1 f(t+iy) e^{-2\pi i n t} dt.$$

When we define

$$g_n : z \mapsto \int_0^1 f(t+z) e^{-2\pi i n t} dt,$$

we can rewrite $a_n(y) = g_n(iy)$. And, g_n is holomorphic : indeed, recall that the well-know implication "h is holomorphic $\Rightarrow$ the integral of h on a closed path vanishes" is actually an equivalence thanks to Morera's Theorem. Thus, the holomorphy of $z \mapsto f(t+z)e^{-2\pi i n t}$ and Fubini's Theorem conclude. Now, thanks to the periodicity, we compute for all $x \in \mathbb{R}$,

$$\begin{aligned} g_n(x) &= \int_0^1 f(t+x) e^{-2\pi i n t} dt = \int_x^{x+1} f(t) e^{-2\pi i n (t-x)} dt \\ &= e^{2\pi i n x} \int_x^{x+1} f(t) e^{-2\pi i n t} dt = e^{2\pi i n x} \int_0^1 f(t) e^{-2\pi i n t} dt = e^{2\pi i n x} g_n(0). \end{aligned}$$

Thus, thanks to the analytic continuation Theorem, $g_n(z) = e^{2\pi i n z} g_n(0)$ for every complex z. We can finally rewrite, when $\tau = x + iy$,

$$f(\tau) = f_y(x) = \sum_{n \in \mathbb{Z}} a_n(y) e^{2\pi i n x} = \sum_{n \in \mathbb{Z}} g_n(0) e^{2\pi i n \tau}$$

$\square$

We now give another proof for Corollary 1.8. The reason we dispense this alternative argument is that it is phrased in terms of differential operators. The latter become compatible with the frameworks of Lie groups acting on their quotients, which is the setting in which some generalized notions of modular forms are studied.

Proof. Let D be the differential operator $D = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$. We see complex functions as complex-valued applications of two real variables here. The Cauchy-Riemann equations give $Df = 0$. Now, as seen before, $f_y : x \mapsto f(x+iy)$ is periodic so we get a Fourier expansion : for all $x + iy \in \mathbb{H}$,

$$f(x+iy) = \sum_{n \in \mathbb{Z}} a_n(y) e^{2\pi i n x},\ a_n(y) = \int_0^1 f(t+iy) e^{-2\pi i n t} dt.$$

Now, the a_n's are differentiable thanks to Liebniz's integral rule :

$$a_n'(y) = \int_0^1 \frac{\partial}{\partial y} f(t+iy)e^{-2\pi int}dt.$$

Also, for every $y \in \mathbb{R}$, f_y is C^2 so integrating by parts gives a bound $|a_n(y)| \leq C(y)/n^2$ where $C(y) = \int_0^1 |\frac{\partial^2}{\partial x^2} f_y|$. In particular, the $|a_n(y)|$'s are uniformly bounded by a C/n^2 on every compact subset of $\mathbb{H}$. Therefore, the Fourier series of f uniformly converges to f on compact sets. Now, as f is actually infinitely differentiable, we can use the same argument with $\sum_n \frac{\partial}{\partial x} a_n(y)e^{2\pi inx}$ and $\sum_n \frac{\partial}{\partial y} a_n(y)e^{2\pi inx}$. This shows that we can permute the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ with the infinite sum to get

$$\begin{aligned} 0 = Df &= \sum_{n\in\mathbb{Z}} \frac{\partial}{\partial x} a_n(y)e^{2\pi inx} + i\frac{\partial}{\partial y} a_n(y)e^{2\pi inx} \\ &= \sum_{n\in\mathbb{Z}} a_n(y)2\pi ine^{2\pi inx} + ia_n'(y)e^{2\pi inx} \\ &= \sum_{n\in\mathbb{Z}} i[a_n(y)2\pi n + a_n'(y)]e^{2\pi inx}. \end{aligned}$$

Identifying the Fourier coefficients yields for all n :

$$a_n'(y) = -2\pi ina_n(y) \Rightarrow a_n(y) = f_n e^{-2\pi iny} \text{ for some constant } f_n.$$

We can now conclude as in the previous proof.

□

We can now rigorously state the definition of a modular form on $SL(2,\mathbb{Z})$.

Definition 1.9. A function $f : \mathbb{H} \to \mathbb{C}$ that is holomorphic and weakly modular is said to be *holomorphic at* ∞ if its Fourier expansion $f(\tau) = \sum_n f_n e^{2\pi in\tau}$ satisfies $f_n = 0$ for all $n < 0$.

Definition 1.10. Let k be an integer. A *modular form of weight* k is a function $f : \mathbb{H} \to \mathbb{C}$ that is

(i) holomorphic on $\mathbb{H}$,

(ii) holomorphic at ∞,

(iii) weakly modular of weight k on $SL(2,\mathbb{Z})$.

Definition 1.11. A modular form f whose Fourier expansion $f(\tau) = \sum_n f_n e^{2\pi in\tau}$ satisfies $f_0 = 0$ is called a *cusp form*.

Proposition 1.12. *We have the elementary properties :*

(i) The collection of all the modular forms of weight k for a fixed integer k form a $\mathbb{C}$-vector space. We denote it $\mathcal{M}_k(SL(2,\mathbb{Z}))$.

(ii) The cusp forms of weight k form a sub-vector space of $\mathcal{M}_k(SL(2,\mathbb{Z}))$. We denote it $\mathcal{S}_k(SL(2,\mathbb{Z}))$.

(iii) The product of a modular form of weight k and one of weight l yields a modular form of weight $k+l$. We get a graded ring

$$\mathcal{M}(SL(2,\mathbb{Z})) = \textstyle\bigoplus_k \mathcal{M}_k(SL(2,\mathbb{Z})).$$

(iv) The cusps forms $\mathcal{S}(SL(2,\mathbb{Z})) = \bigoplus_k \mathcal{S}_k(SL(2,\mathbb{Z}))$ form an ideal in $\mathcal{M}(SL(2,\mathbb{Z}))$.

Proof. All of these are easy verifications. To check the point (iv), recall that the multiplication of two absolutely converging Fourier series $\sum_n a_n e^{2\pi i n\tau}$ and $\sum_n b_n e^{2\pi i n\tau}$ is given by $\sum_n c_n e^{2\pi i n\tau}$ where

$$c_n = \sum_{i+j=n} a_i b_j.$$

□

We have the trivial examples :

Example 1.13. Any constant function $f : \tau \mapsto c$ is a modular form of weight 0. More generally, the modular forms of weight 0 are the $SL(2,\mathbb{Z})$-invariant holomorphic functions.

Example 1.14. $\mathcal{M}_k(SL(2,\mathbb{Z})) = \{0\}$ whenever k is odd. Indeed, $-Id$ lies in $SL(2,\mathbb{Z})$ and acts trivially on $\mathbb{H}$. Thus, for all $\tau \in \mathbb{H}$ and $f \in \mathcal{M}_k(SL(2,\mathbb{Z}))$, $f(\tau) = f(-Id\tau) = (-1)^k f(\tau) = -f(\tau)$ which implies $f = 0$.

We conclude this section with a discussion about the holomorphy at ∞ (and why it is named as it is). Let $\tau = x + iy \in \mathbb{H}$. Since $y > 0$,

$$e^{2\pi i\tau} = \underbrace{e^{2\pi i x}}_{\in S^1} \underbrace{e^{-2\pi y}}_{\in]0,1[} \in D = \{z \in \mathbb{C}, |z| < 1\}.$$

And, we can reach any point of $D' = D \setminus \{0\}$ via $\mathbb{H} \ni \tau \mapsto e^{2\pi i\tau}$ by writing it as $re^{i\theta}$, $r \in]0,1[$ and choosing the right $x = \frac{\theta}{2\pi}$ and $y = -\frac{ln(r)}{2\pi} > 0$. Moreover, $\tau \mapsto e^{2\pi i\tau}$ is $\mathbb{Z}$-periodic and $e^{2\pi i\tau}$ tends to 0 as $Im(\tau)$ tends to ∞. Visually, the map $\tau \mapsto e^{2\pi i\tau}$ contracts the upper half plane onto an infinite strip $\mathbb{R}\times]0,1[$ then wraps this strip onto the punctured disc D'. Recall now that the complex exponential function can be locally holomorphically inverted around any non-zero complex number. And, any two possible inverses differ by an element of $2\pi i\mathbb{Z}$. f being $\mathbb{Z}$-periodic, the map

$$\begin{aligned} g : D' &\longrightarrow \mathbb{C}; \\ q &\longrightarrow f\left(\frac{ln(q)}{2\pi i}\right) \end{aligned}$$

is well-defined. It is holomorphic because the logarithm can be expressed holomorphically around each point of the punctured disc. We also have by construction $f(\tau) = g(e^{2\pi i\tau})$. Considering the Laurent expansion of g at 0 yields

$$g(q) = \sum_{n\in\mathbb{Z}} g_n q^n.$$

Evaluating at $q = e^{2\pi i\tau}$ gives

$$\sum_{n\in\mathbb{Z}} g_n e^{2\pi i n\tau} = \sum_{n\in\mathbb{Z}} g_n (e^{2\pi i\tau})^n = g(e^{2\pi i\tau}) = f(\tau),$$

so the g_n's are the Fourier coefficients of f. We can now explain the terminology "holomorphic at ∞" : demanding $g_n = 0$ for every negative n is equivalent to requiring that g can be holomorphically continued at 0. These considerations enable us to get a new criterion :

Corollary 1.15. *Let $f : \mathbb{H} \to \mathbb{C}$ be holomorphic and weakly modular. Then,*

$$\begin{aligned} f \text{ is holomorphic at } \infty &\Leftrightarrow \lim_{Im(\tau)\to\infty} f(\tau) \text{ exists} \\ &\Leftrightarrow f(\tau) \text{ is bounded as } Im(\tau) \to \infty. \end{aligned}$$

Moreover, in that case,

$$f \text{ is a cusp form } \Leftrightarrow \lim_{Im(\tau)\to\infty} f(\tau) = 0.$$

Proof. If f is a modular form, since $Im(\tau) \to \infty \Leftrightarrow e^{2\pi i \tau} \to 0$,

$$\lim_{Im(\tau)\to\infty} f(\tau) = \lim_{Im(\tau)\to\infty} g(e^{2\pi i \tau}) = \lim_{z\to 0} g(z) = g_0.$$

Thus, the limit exists and it is the first Fourier coefficient of f. This gives the characterization of cusp forms. If we suppose that the limit exists, $f(\tau)$ is of course bounded as $Im(\tau)$ tends to infinity. Finally, if one supposes that $f(\tau)$ is bounded as $Im(\tau)$ tends to infinity, then $g(q)$ is bounded as q tends to 0 which can only happen if the negative coefficients in $\sum_n g_n q^n$ vanish.

□

2. First Examples of Modular Forms and Cusp Forms

We introduce in this second part an important non-trivial example of modular forms : the Eisenstein series.

Definition 2.1. We define the *Eisenstein series of weight* k for every even integer $k > 2$ as the series

$$G_k(\tau) := \sum_{(n,m)\in L} \frac{1}{(n\tau+m)^k}, \text{ for all } \tau \in \mathbb{H},\ L := \mathbb{Z}^2 \setminus \{(0,0)\}.$$

This is the first example of non-trivial modular form of weight $k \neq 0$. In order to prove this, we start by checking that this is an actual holomorphic function on $\mathbb{H}$. We define the following subsets of $\mathbb{H}$: for any two strictly positive real numbers α and β, we set

$$\Omega_\beta^\alpha := \{z = x + iy \in \mathbb{Z},\ x \leqslant \alpha,\ y \geqslant \beta\}.$$

Lemma 2.2. *For all α and β as above, $k > 2$ an even integer, G_k converges absolutely and uniformly on Ω_β^α.*

Proof. We use the Weierstrass M-test to get the absolute and uniform convergence. Recall its statement :

Theorem 2.3 (Weierstrass M-test)**.** *Let $\{fn\}$ be a sequence of complex functions defined on a set Ω. Suppose that for all $n \in \mathbb{N}$, there is a constant c_n such that $|f_n(x)| \leqslant c_n$ on all Ω. If the sum $\sum\limits_{n=1}^{\infty} c_n$ converges, then the sum $\sum\limits_{n=1}^{\infty} fn(x)$ converges absolutely and uniformly on Ω.*

To apply the Theorem, we first prove the existence of a $r > 0$ satisfying

$$|n\tau + m| \geqslant r max\{|n|, |m|\} \text{ for all } \tau \in \Omega_\beta^\alpha \text{ and } (n,m) \in L.$$

We claim that there is a constant $1 > r' > 0$ such that for all $\delta \in \mathbb{R}$ and all $\tau = x + iy \in \mathbb{H}$, $|\tau + \delta| \geqslant r' max\{1, |\delta|\}$. Indeed, if $|\delta| < 1$,

$$|\tau + \delta|^2 = (x+\delta)^2 + y^2 \geqslant y^2 \geqslant \beta^2 = \beta^2 max\{1, |\delta|\}^2.$$

Now, if $1 \leqslant |\delta| \leqslant 3\alpha$ and $y = im(\tau) > \alpha$

$$|\tau + \delta|^2 = (x+\delta)^2 + y^2 \geqslant y^2 \geqslant \alpha^2 > (\frac{1}{3}\delta)^2 = (\frac{1}{3} max\{1, |\delta|\})^2.$$

If $y = Im(\tau) \leqslant \alpha$, we must have $\alpha \leqslant \beta$. Thus we have that the point (δ, τ) is in the compact set $A := [1, 3\alpha] \times \{z \in \Omega_\beta^\alpha | \beta \leqslant Im(z) \leqslant \alpha\}$. Since this set is compact, and that the continuous function $(\delta, z) \mapsto \frac{|\delta+z|}{|\delta|}$ is well-defined on A, this mapping admits a minimum $M \in A$. Therefore

$$|(\delta, \tau)| \geqslant M|\delta| = M max\{1, |\delta|\}.$$

The last case is $|\delta| > 3\alpha$ and $|\delta| \geqslant 1$, but then

$$|\tau + \delta| \geqslant |\delta| - \alpha.$$

Thus, since $|\delta| > 3\alpha$, we have $|\delta| - \alpha > \frac{2}{3}|\delta| = \frac{2}{3} max\{1, |\delta|\}$. Therefore choosing $r' = min\left\{\frac{1}{3}, M, \beta\right\}$ yields the desired result. We can now deduce our first claim. Let $(m,n) \in L$, assume first that $n \neq 0$, then,

$$|n\tau + m| = \left|\tau + \frac{m}{n}\right| |n| \geqslant r' max\left\{1, \left|\frac{m}{n}\right|\right\} |m| = r' max\{|m|, |n|\}.$$

And if $n = 0$ then, clearly $|m| \geqslant max\{|m|, |n|\} > r'max\{|m|, |n|\}$. Thus $r = r'$ is adequate. We get the bound, for all $\tau \in \Omega_\beta^\alpha$,

$$\sum_{(n,m)\in L} \left| \frac{1}{(n\tau + m)^k} \right| \leqslant \sum_{(n,m)\in L} \frac{1}{r^k max\{|m|, |n|\}^k}.$$

Now, we can deduce the absolute convergence of G_k as follows. First note that

$$\sum_{(n,m)\in L} \frac{1}{r^k max\{|m|, |n|\}^k} = \sum_{j=1}^{\infty} \frac{s(j)}{r^k j^k},$$

where $s(j) := |\{(n, m) \in L \mid max\{|m|, |n|\} = j\}|$. But one easily checks that $s(j) = 8j$ so the last sum becomes $\sum_j 8r^{-k} j^{-(k+1)}$, which converges as $k > 2$. Finally, the Weierstrass M-test gives the absolute and uniform convergence of G_k on every Ω_β^α.

$\square$

Since each compact subset of $\mathbb{H}$ is contained in a Ω_β^α, the series converges absolutely and uniformly on all compact subsets of $\mathbb{H}$. Thus, Morera's Theorem gives the holomorphy of G_k on $\mathbb{H}$. Moreover, the absolute convergence allows us to permute the terms of G_k and prove the modularity of the Eisenstein series.

Theorem 2.4. *G_k is a modular form of weight k for al $k > 2$ even.*

Proof. Indeed, for all $\tau \in \mathbb{H}$,

$$G_k(\tau + 1) = \sum_{(n,m)\in L} \frac{1}{(n\tau + m + n)^k}.$$

But we can see that this is exactly $G_k(\tau)$ since $(n, m + n)$ runs over all L when (n, m) does. We now compute $G_k(-1/\tau)$:

$$\begin{aligned} G_k(-1/\tau) &= \sum_{(n,m)\in L} \frac{1}{(-n\frac{1}{\tau} + m)^k} = \sum_{(n,m)\in L} \frac{\tau^k}{(\tau(-n\frac{1}{\tau} + m))^k} \\ &= \tau^k \sum_{(n,m)\in L} \frac{1}{(-n + m\tau)^k} = \tau^k G_k(\tau). \end{aligned}$$

Similarly, $(m, -n)$ runs over L when (n, m) does.

We now check that G_k is holomorphic at infinity. If $\tau \in \Omega_\beta^\alpha$, τ still lies in Ω_β^α as $Im(\tau)$ tends to infinity. Therefore, we can permute the limits to get

$$\begin{aligned} \lim_{Im(\tau)\to\infty} G_k(\tau) &= \lim_{Im(\tau)\to\infty} \sum_{\substack{(n,m)\in L \\ n\neq 0}} \frac{1}{(n\tau + m)^k} + 2\zeta(k) \\ &= \sum_{\substack{(n,m)\in L \\ n\neq 0}} \lim_{Im(\tau)\to\infty} \frac{1}{(n\tau + m)^k} + 2\zeta(k) = 2\zeta(k). \end{aligned}$$

Thus G_k is holomorphic at infinity. We proved that G_k is a modular form of weight k for any even $k > 2$. $\square$

We now compute the Fourier coefficients of G_k. We will need the following formula.

Lemma 2.5 (Lipschitz's formula)**.** *Let $k \geqslant 2$ and $\tau \in \mathbb{H}$. Then*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n\tau}.$$

Proof. The cotangent can be expanded as the series

$$\pi cot(\pi\tau) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{\tau-n} + \frac{1}{\tau+n} \right).$$

Moreover, we have the following identities:

$$cot(\tau\pi) = \frac{cos(\tau\pi)}{sin(\tau\pi)} = i\frac{e^{\pi i\tau}+e^{-\pi i\tau}}{e^{\pi i\tau}-e^{-\pi i\tau}} = i\frac{q+1}{q-1} = -i\frac{1+q}{1-q} = -i(1+q)\sum_{n=0}^{\infty} q^n = -i(1+2\sum_{n=1}^{\infty} q^n),$$

where $q = e^{2\pi i\tau}$. Therefore,

$$\frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{\tau-n} + \frac{1}{\tau+n} \right) = -2\pi i \left(\frac{1}{2} + \sum_{n=1}^{\infty} q^n \right).$$

By differentiating $k-1$ times this equation, we get :

$$\begin{aligned}
(-1)^{k-1}\frac{(k-1)!}{\tau^k} + \sum_{n=1}^{\infty} &\left(\frac{(-1)^{k-1}(k-1)!}{(\tau-n)^k} + \frac{(-1)^{k-1}(k-1)!}{(\tau+n)^k} \right) \\
&= (-1)^{k-1}\frac{(k-1)!}{\tau^k} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(\frac{(-1)^{k-1}(k-1)!}{(\tau+n)^k} \right) \\
&= \sum_{n\in\mathbb{Z}} \left(\frac{(-1)^{k-1}(k-1)!}{(\tau+n)^k} \right) \\
&= -2\pi i \sum_{n=1}^{\infty} (2\pi i n)^{k-1} q^n \\
&= -(2\pi i)^k \sum_{n=1}^{\infty} n^{k-1} q^n.
\end{aligned}$$

Divide both sides by $(-1)^{k-1}(k-1)!$ yields the desired

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{-(2\pi i)^k}{(-1)^{k-1}(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^n = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} n^{k-1} q^n.$$

□

We can now deduce the Fourier expansion of G_k. Let $\tau \in \mathbb{H}$.

$$\begin{aligned}
G_k(\tau) = \sum_{(n,m)\in L} \frac{1}{(n\tau+m)^k} &= 2\zeta(k) + \sum_{\substack{(n,m)\in\mathbb{Z}^2 \\ n\neq 0}} \frac{1}{(n\tau+m)^k} \\
&= 2\zeta(k) + \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} \frac{1}{(n\tau+m)^k} + (-1)^k \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} \frac{1}{(n\tau+m)^k} \\
&= 2\zeta(k) + 2\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} \frac{1}{(n\tau+m)^k} \qquad (3)
\end{aligned}$$

Now applying the Lipschitz's formula to $\sum_{m\in\mathbb{Z}} \frac{1}{(n\tau+m)^k}$ for each n yields

$$\begin{aligned} G_k(\tau) &= 2\zeta(k) + 2\sum_{n=1}^{\infty} \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} e^{2\pi i m n\tau} \\ &= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m^{k-1} e^{2\pi i m n\tau} \\ &= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n\tau}, \text{ where } \sigma_k(n) = \sum_{\substack{m|n \\ m>0}} m^k. \end{aligned}$$

We use this to define an analogue to the Eisenstein series but for weight $k = 2$.

Definition 2.6. We define the *Eisenstein series of weight 2* as the sum

$$G_2(\tau) := 2\zeta(2) + 2\sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} \frac{1}{(n\tau+m)^2} \tag{4}$$

We can not use the proof of the convergence of G_k to prove that G_2 converges. In fact this series is not absolutely convergent, therefore one can not change the order of summation. However, the Lipschitz's formula holds for $k = 2$. Since the sum of G_2 is already of the form of (3), we still have an expansion

$$G_2(\tau) = \ 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n\tau}.$$

The sum in this expression converges to a holomorphic function. Since we cannot permute the terms of the sum in (4), we can not prove the modularity of G_2 as we did for the other Eisenstein series. Actually, G_2 is not modular and we have the following result :

Lemma 2.7. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$*, and* $\tau \in \mathbb{H}$*. Then,*

$$G_2(\gamma\tau) = j(\gamma,\tau)^2 G_2(\tau) + \pi i c j(\gamma,\tau).$$

Definition 2.8. We define the *normalized Eisenstein series* as $E_k := \dfrac{1}{2\zeta(k)} G_k$.

We now use the Einsenstein series and the stability under sums and products of modular forms to define our first cusp form.

Definition 2.9. We define the *discrimimant function* Δ as

$$\begin{aligned} \Delta : \mathbb{H} &\longrightarrow \mathbb{C} \\ \tau &\longmapsto \frac{1}{1728}(E_4^3 - E_6^2). \end{aligned}$$

The discriminant function is a modular form of weight $k = 12$ thank to Proposition 1.12. Moreover, it is a cusp form because, using Corollary 1.15,

$$\lim_{Im(\tau)\to\infty} E_k(\tau) = 1 \Rightarrow \lim_{Im(\tau)\to\infty} \Delta(\tau) = 0.$$

We now prove that Δ is a non-trivial cusp form by computing its first Fourier coefficient. Using the usual product rule for a Fourier series $\sum_n a_n q^n$, we can identify the first coefficients of $\left(\sum_n a_n q^n\right)^2$ as

$2a_0a_1$ and the first one of $\left(\sum_n a_nq^n\right)^2$ as $3a_0^2a_1$. Therefore, the first coefficient of E_6^2 is $2\frac{1}{\zeta(6)}\frac{(2\pi i)^6}{(5)!}\sigma_5(1)$ For E_4^3, it is $3\frac{1}{\zeta(4)}\frac{(2\pi i)^4}{(3)!}\sigma_3(1)$. Note also that $\sigma_k(1)=1$ for all k so the first coefficient b_1 of Δ satisfies

$$\begin{aligned}1728b_1 &= 2\frac{1}{\zeta(6)}\frac{(2\pi i)^6}{5!} - 3\frac{1}{\zeta(4)}\frac{(2\pi i)^4}{3!}\\ &= 2\frac{945}{\pi^6}\frac{(2\pi i)^6}{120} - 3\frac{90}{\pi^4}\frac{(2\pi i)^4}{6}\\ &= -1728.\end{aligned}$$

Therefore $b_1=-1$ and thus Δ is a non-trivial cusp form. It will be useful to express Δ as follows.

Lemma 2.10. *The discriminant function can be expressed as the product*

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24},\ \textit{where } q = e^{2\pi i\tau}.$$

We can see that this implies in particular that for all $\tau\in\mathbb{H}$, $\Delta(\tau)\neq 0$. This definition enables us to compute the derivative of Δ.

Lemma 2.11. *The discriminant function and E_2 satisfy the relation*

$$\Delta' = 2\pi i\Delta E_2.$$

Proof. Since $\Delta(\tau)\neq 0$, we can define a invert to the complex exponential around $\Delta(\tau)$. The inverse we obtain, ln, is only well-defined up to an element of $2\pi i\mathbb{Z}$. In particular, any two inverses differ by a constant, so the derivative of $ln\circ\Delta$ is well-defined. We compute :

$$\begin{aligned}(ln\circ\Delta)'(\tau) =& \frac{d}{d\tau}\left(ln\left(q\prod_{n=1}^{\infty}(1-q^n)^{24}\right)\right)\\ =&2\pi i + \sum_{n=1}^{\infty}24\frac{d}{d\tau}ln(1-q^n)\\ =&2\pi i - \sum_{n=1}^{\infty}48\pi in\frac{q^n}{1-q^n}\\ =&2\pi i - 48\pi i\sum_{n=1}^{\infty}nq^n\sum_{m=0}^{\infty}q^{nm}\\ =&2\pi i - 48\pi i\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}nq^{nm}\\ =&2\pi i\left(1-\frac{8\pi^2}{2\zeta(2)}\sum_{n=1}^{\infty}\sigma_1(n)e^{2\pi in\tau}\right) = 2\pi iE_2.\end{aligned}$$

Since $(ln\circ\Delta)' = \frac{\Delta'}{\Delta}$ we can conclude.

$\square$

To conclude, we produced a non-trivial modular form G_k of weight k for every even positive integer $k\neq 2$. We also previously proved that odd weights do not admit non-trivial modular forms. Finally, we constructed a non-trivial cusp form of weight 12, which turns out to be the non-trivial cusp form of lowest weight.

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