The Valence Formula and Its Applications

Carl Felix Waller and Ningjie Tan

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1 The Space of Modular Forms

Let f be a meromorphic non-zero function on the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and let $p \in \mathbb{H}$. We denote by $v_p(f)$ *the order of* f *at* p, so $v_p(f)$ is the integer n such that $\frac{f(z)}{(z-p)^n}$ is holomorphic and non-zero at p. Observe that for f a weakly modular function of weight $2k$, [1] we have $v_p(f) = v_{\gamma p}(f)$ for all $p \in \mathbb{H}$ and $\gamma \in \bar{\Gamma} := \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$. Indeed, since

$$f(\gamma.z) = (cz + d)^{2k} f(z)$$

we see that the term $(cz + d)^{2k}$ will never cancel a pole or a zero since $z \in \mathbb{H}$ and $c, d \in \mathbb{R}$. Thus $v_p(f)$ only depends on the image of $p \in \mathbb{H}$ in $\mathbb{H}/\bar{\Gamma}$. We will, by slight abuse of notation, write p for the element in the quotient, hoping that it will always be clear from the context what is meant. Recall that for f a 1-periodic meromorphic function on $\mathbb{H}$ we can define a function g on $\mathbb{D}^\times$ via $g(q) = f(\frac{\log(q)}{2\pi i})$. This functions is meromorphic on the punctured disk [2], recall the definition.

Definition 1.1. A 1-periodic meromorphic function f on $\mathbb{H}$ is called *$meromorphic\ at\ infinity$* if g is meromorphic at zero.

Indeed, one can easily verify that this definition is equivalent to the definition we had in the second talk. Then for f a modular function simply set $v_\infty(f) := v_0(g)$.

Claim 1.2. *Let f be a modular function then f has finitely many poles and zeros in $\mathbb{H}/\bar{\Gamma}$*

Proof. Since f is meromorphic at ∞ there exists an $1 > R > 0$ such that

$$g(q) = \sum_{n=v_\infty(f)}^{\infty} a_n q^n$$

converges for $0 < |q| < R$. Moreover let us assume that g has no zeros for $0 < |q| < R$, which we can do since the zeros of a holomorphic function are always isolated. Now recall that $f(z) = g(e^{2\pi i z})$ and the preimage of $q \in \mathbb{D}^\times$ under the map $\mathbb{H} \to \mathbb{D}^\times,\ z \mapsto e^{2\pi i z}$ is

$$z(q) = -\frac{i}{2\pi} \log(q) = -\frac{i}{2\pi} \log|q| + \frac{\arg(q)}{2\pi}, \tag{1}$$

[1] We saw last time that there are non-zero weakly modular function with odd weight, that is why we will use the $2k$ notation in our talk.

[2] Indeed, since log is holomorphic in a neighbourhood of any point in $\mathbb{D}^\times$ this follows from the fact that the composition $f_1 \circ f_2$ of a meromorphic function f_1 and a holomorphic and non-constant function f_2 is always meromorphic.

where arg and the first log denote the usual multivalued functions. Thus g having no poles and zeros inside $0 < |q| < R$ is equivalent to f having no poles and zeros for $\infty > \mathrm{Im}(z) > -\frac{\log(R)}{2\pi}$. Thus f has no poles and zeros for $\mathrm{Im}(z) > -\frac{\log(R)}{2\pi}$ and since the intersection of the fundamental domain $F := \{z \in \mathbb{C} : -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}, |z| \geq 1\}$ with $\{\mathrm{Im}(z) \leq -\frac{\log(R)}{2\pi}\}$ is compact and contains all zeros and poles of f the claim follows. $\square$

Finally, let us denote by e_p the order of the stabilizer of $p \in \mathbb{H}$ under the action of $\bar{\Gamma}$ on $\mathbb{H}$, that is $e_p = 2$ if p is congruent modulo $\bar{\Gamma}$ to i, $e_p = 3$ if p is congruent modulo $\bar{\Gamma}$ to $\rho := e^{\pi i/3}$ and else $e_p = 1$ (see Corollary 1 from the first talk). Let us denote the quotient of the action of $\bar{\Gamma}$ on $\mathbb{H}$ by $\mathbb{H}/\bar{\Gamma}$. Now we can prove the valence formula.

Theorem 1.3. *Let f be a modular function of weight $2k$ not identically zero then*

$$v_\infty(f) + \sum_{p \in \mathbb{H}/G} \frac{1}{e_p} v_p(f) = \frac{k}{6} \tag{2}$$

or equivalently

$$v_\infty(f) + \frac{1}{3} v_\rho(f) + \frac{1}{2} v_i(f) + \sum_{p \in \mathbb{H}/\bar{\Gamma}}^{*} v_p(f) = \frac{k}{6}, \tag{3}$$

where the stared sum means that we are not summing over i and ρ.

Proof. We will use the logarithmic integral around some contour to conclude the result. First, let us assume that f has no poles or zeros on ∂F except for maybe at ρ, i or $\rho^2 = S\rho$, integrate around the contour $\mathcal{C}$ in Figure 1 and let the radius $\epsilon > 0$ of arcs around i, ρ, ρ^2 go to zero.

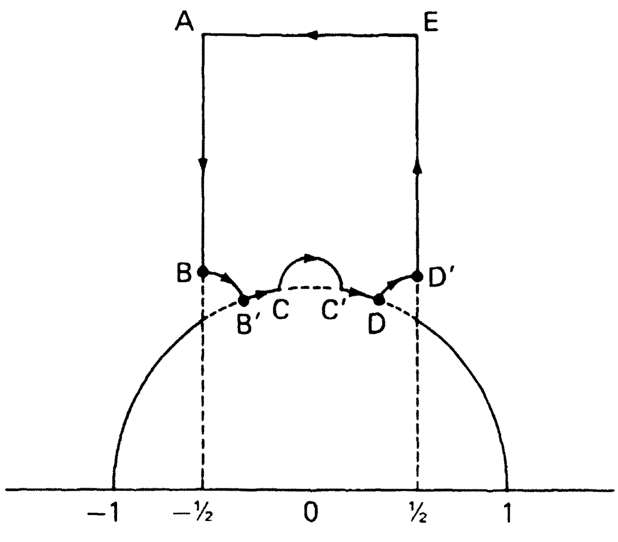

Figure 1: Contour $\mathcal{C}$ taken from [1]

On the one hand

$$\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{df}{f}\overset{\epsilon\to 0}{\to}\sum_{p\in\mathbb{H}/\bar{\Gamma}}^{*}\frac{1}{e_p}v_p(f) \tag{4}$$

Conversely let us evaluate each line segment separately, the arc from B' to C we evaluate using a change of variable $z\mapsto S(z)=-\frac{1}{z}$

$$\begin{aligned}\int_{B'}^{C}\frac{f'(z)}{f(z)}dz &= \int_{D}^{C'}\frac{f'(Sz)}{f(Sz)}S'(z)dz\\ &= \int_{D}^{C'}\left[\frac{2k}{z}+\frac{f'(z)}{f(z)}\right]dz,\end{aligned}$$

where we used that

$$f'(Sz)S'(z)=(f\circ S)'(z)=\frac{d}{dz}f(Sz)=\frac{d}{dz}z^{2k}f(z)=2kz^{2k-1}f(z)+z^{2k}f'(z).$$

The second term cancels with the other arc from C' to D and the remaining integral we can explicitly evaluate by choosing the path $\gamma(t)=e^{it}$ that is

$$\begin{aligned}\frac{1}{2\pi i}\int_{B'}^{C}\frac{f'(z)}{f(z)}dz+\frac{1}{2\pi i}\int_{C'}^{D}\frac{f'(z)}{f(z)}dz &= \frac{1}{2\pi i}\int_{D}^{C'}\frac{2k}{z}dz\\ \overset{\epsilon\to 0}{\to}\frac{1}{2\pi i}\int_{\pi/3}^{\pi/2}\frac{2k}{e^{it}}\,ie^{it}dt &= \frac{k}{\pi i}\left[\frac{\pi}{2}-\frac{\pi}{3}\right]=\frac{k}{6}.\end{aligned} \tag{5}$$

The integrals around i,ρ,ρ^2 are easily computed

$$\frac{1}{2\pi i}\int_{B}^{B'}\frac{df}{f}\overset{\epsilon\to 0}{\to}-\frac{1}{6}v_\rho(f) \tag{6}$$

$$\frac{1}{2\pi i}\int_{D}^{D'}\frac{df}{f}\overset{\epsilon\to 0}{\to}-\frac{1}{6}v_\rho(f) \tag{7}$$

$$\frac{1}{2\pi i}\int_{C}^{C'}\frac{df}{f}\overset{\epsilon\to 0}{\to}-\frac{1}{2}v_i(f). \tag{8}$$

Now we will show that $\frac{1}{2\pi i}\int_E^A\frac{df}{f}=-v_\infty(f)$. It requires a little more thought[3] than the previous ones. Again using the fact that poles and zeros are isolated we can find an open horizontal strip $U\subset\mathbb{H}$ about the line from E to A, which contains all $z\in\mathbb{H}$ with imaginary part large enough. Set $h=f'/f$ this is again 1-periodic and holomorphic on the strip U. Last time we proved that a holomorphic 1-periodic map on $\mathbb{H}$ can always be written in a Fourier series. Looking at the proof it is easy to see that the same will hold for a 1-periodic function that is only holomorphic on a horizontal strip. Hence, we can write

$$f(z)=\sum_{n=v_\infty(f)}^{\infty}f_ne^{2\pi inz},\quad h(z)=\sum_{n\in\mathbb{Z}}h_ne^{2\pi inz},\quad\forall z\in U.$$

[3]It is also possible to evaluate this integral by changing variable $z\mapsto e^{2\pi iz}$, however then there is a subtlety with the inverse not being continuous, which needs to be dealt with.

Since the series is absolutely convergent we can interchange the sum with the integral and get that

$$\int_E^A \frac{f'(z)}{f(z)}dz = \sum_{n\in\mathbb{Z}} \int_E^A h_n e^{2\pi i n z} dz = \sum_{n\in\mathbb{Z}} \int_{1/2}^{-1/2} h_n R e^{2\pi i n t} dt = h_0$$

where we simply parameterize the line integral as $z(t) = t - i\frac{\log R}{2\pi}$. Conversely observe that f' also has a Fourier expansion with coefficients $f'_n = 2\pi i n f_n$. Also we have

$$\lim_{\mathrm{Im}(z)\to\infty} h(z) = \lim_{\mathrm{Im}(z)\to\infty} \frac{\sum_{n=v_\infty(f)}^{\infty} f'_n e^{2\pi i n z}}{\sum_{n=v_\infty(f)}^{\infty} f_n e^{2\pi i n z}} = \frac{f'_{v_\infty(f)}}{f_{v_\infty(f)}} = 2\pi i v_\infty(f)$$

which is finite. Thus by Corollary 1.15 from the second talk, h is holomorphic at infinity and moreover $h_0 = 2\pi i v_\infty(f)$, we conclude that

$$\frac{1}{2\pi i}\int_E^A \frac{f'(z)}{f(z)}dz = -v_\infty(f). \tag{9}$$

Lastly integrating along the left hand vertical line we perform a change of variable $z \mapsto z+1 = T(z)$

$$\begin{aligned}\int_A^B \frac{f'(z)}{f(z)}dz &= \int_E^{D'} \frac{f'(z+1)}{f(z+1)}T'(z)dz \\ &= \int_E^{D'} \frac{f'(z)}{f(z)}dz = -\int_{D'}^E \frac{f'(z)}{f(z)}dz\end{aligned}$$

thus the two vertical lines cancel each other.
Combining equations (4)-(9) gives the desired formula. Recall that we still need to show the theorem for the general case, where we allow zeros and poles to be anywhere on the boundary ∂F. So let $\lambda \in \partial F \setminus \{i, \rho, \rho^2\}$ be a pole or zero of f, observe that there then will also be a second zero or pole $\lambda' \in \bar{\Gamma}\lambda$ in the orbit of λ. It doesn't really matter how we integrate around λ and λ' but for convenience we choose paths such that equation (4) still holds, that is only one point of the two is inside the contour. If λ and λ' are on the vertical lines, then the two arcs around λ and λ' cancel just as before. For λ on the unit circle the two terms will also cancel, indeed performing the same change of variable $z \mapsto Sz$

$$\int_\lambda \frac{df}{f} = -\int_{\lambda'} \Big[\frac{2k}{z} + \frac{f'(z)}{f(z)}\Big]dz$$

where $\int_\lambda$ denotes the positively oriented arc around λ as in Figure 2. The first term goes to zero as we let the radius of the half circle go to zero, the second term cancels the arc around λ'.

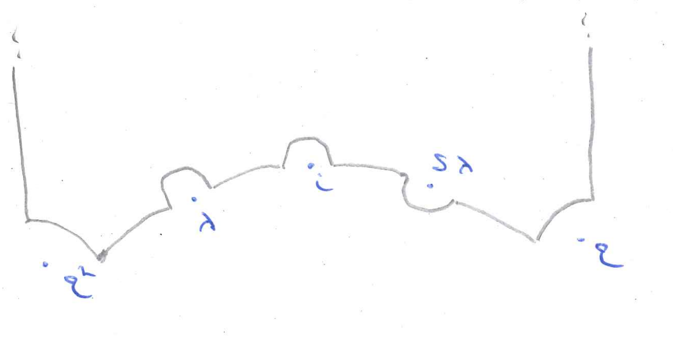

Figure 2: Modified contour for extremal point on the unit disk

Clearly we can apply the same argument for multiple poles or zeros on the boundary (recall that there can only be finitely many). This completes the proof. $\square$

We now discuss the space of modular forms. For $k \in \mathbb{Z}$ denote M_{2k}, S_{2k} the $\mathbb{C}$-vector space of modular forms, respectively cusp forms of weight $2k$. For $k \geq 2$ denote $G_{2k} \in M_{2k}$ the Eisenstein series of weight $2k$, which were defined in the second talk. We can think of S_{2k} as the kernel of the map

$$\begin{aligned} M_{2k} &\to \mathbb{C} \\ f &\mapsto f(\infty). \end{aligned}$$

This is a linear map between complex vector spaces, by linear algebra $\dim(M_{2k}) - \dim(S_{2k}) = \dim(M_{2k}/S_{2k}) \leq \dim \mathbb{C} = 1$. Moreover recall that for $k \geq 2$ we have $G_{2k}(\infty) = 2\zeta(2k) \neq 0$, by the proof of Theorem 2.4 from the last talk. Thus we can write

$$M_{2k} = S_{2k} \oplus \mathbb{C}.G_{2k}, \quad \forall k \geq 2 \tag{10}$$

where $\mathbb{C}.G_{2k}$ denotes the one-dimensional $\mathbb{C}$-vector space with basis G_{2k}. Also recall the definition of the discriminant function $\Delta := \frac{1}{1728}(E_4^3 - E_6^2) \in S_{12}$, where $E_{2k} := \frac{1}{2\zeta(2k)} G_{2k}$ denotes the normalized Eisenstein series of weight $2k$. We proved that Δ is a non-trivial element of S_{12}. Using formula (3) we can prove some results concering the spaces M_{2k}.

Theorem 1.4. *i) We have $M_{2k} = 0$ for $k < 0$ and $k = 1$*

ii) Multiplication by Δ defines an isomorphism from M_{2k-12} to S_{2k}

iii) For $k = 0, 2, 3, 4, 5$ M_{2k} is one-dimensional with basis $1, G_4, G_6, G_8, G_{10}$, so also $S_{2k} = 0$

Proof. Let $k < 0$ or $k = 1$ and let $f \in M_{2k}$. By contradiction assume that f is non-zero, then the we know that Theorem 1.3 holds. That is

$$v_\infty(f) + \sum_{p \in \mathbb{H}/\bar{\Gamma}}^{*} v_p(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\rho(f) = \frac{k}{6}. \tag{11}$$

Since f is holomorphic the left hand side of this equation actually is ≥ 0. Thus we already see that we get a contradiction for $k < 0$. For $k = 1$ observe that this equation can also never hold. For the simple reason that there do not exist non-negative integers (n, n', n'') such that

$$n + \frac{n'}{2} + \frac{n''}{3} = \frac{1}{6}. \tag{12}$$

This proves i). Now we prove ii). Since Δ is a cusp form it has a zero at infinity, that is $v_\infty(\Delta) \geq 1$. For $f = \Delta$ formula (11) holds. Since the right hand side is equal to 1 we get that actually $v_\infty(\Delta) = 1$ and $v_p(\Delta) = 0$ for all $p \in \mathbb{H}$. That is Δ has a simple zero at infinity and is non-zero on $\mathbb{H}$. Now we are ready to define for arbitrary k the linear map

$$\begin{aligned} S_{2k} &\to M_{2k-12} \\ f &\mapsto \frac{f}{\Delta}. \end{aligned}$$

We need to check that this is well defined, i.e. $\frac{f}{\Delta} \in M_{2k-12}$. First, observe that $\frac{f}{\Delta}$ is clearly weakly modular of weight $2k - 12$. Second, $\frac{f}{\Delta}$ has no poles in $\mathbb{H}$ since we just proved that Δ is non-zero on $\mathbb{H}$. Finally, $v_\infty(\frac{f}{\Delta}) = v_\infty(f) - v_\infty(\Delta) = v_\infty(f) - 1 \geq 0$, since f is a cusp from. This proves ii) since the map $M_{2k-12} \to S_{2k}$ given by $g \mapsto \Delta \cdot g$ is clearly well defined, linear and an inverse.
Now let $0 \leq k \leq 5$, then $2k - 12 < 0$ and thus by i) $M_{2k-12} = 0$. By ii) $S_{2k} = 0$ and thus $\dim M_{2k} \leq 1$. For $k = 0, 2, 3, 4, 5$ we know that $1, G_4, G_6, G_8, G_{10}$ are non-zero elements in M_{2k} which shows iii) and completes the proof. □

Corollary 1.5. *We have for $k \geq 0$*

$$\dim M_{2k} = \begin{cases} \lfloor \frac{k}{6} \rfloor & \text{if } k \equiv 1 \bmod 6 \\ \lfloor \frac{k}{6} \rfloor + 1 & \text{else} \end{cases}$$

where $\lfloor\ \rfloor$ denotes the floor function, i.e. for $x \in \mathbb{R}$, $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : n \leq x\}$.

Proof. This holds for $0 \leq k \leq 5$ by part ii) of Theorem 1.4 and combining equation (10) with part iii) of Theorem 1.4 yields the result for all larger k by induction. □

A more concrete application of Theorem 1.4 is the following observation about the normalized Eisenstein series.

Claim 1.6. $E_4^2 = E_8$, $E_4 E_6 = E_{10}$ *and* $E_6 E_8 = E_{14}$

Proof. Clearly $E_4^2 \in M_8$ and since E_{2k} are normalized at infinity $E_4^2 - E_8 \in S_8$. However from Theorem 1.4 we know that $S_8 = 0$, hence we get the first equality. The same argument works for the other two since $S_{10} = S_{14} = 0$. □

Note that the argument in the proof only goes through for these 3 cases. Generally $E_k E_l \neq E_{k+l}$ since we can have non-zero cusp forms. One can use Claim 1.6 to derive relations between the Fourier coefficients of the Eisenstein series. Recall that for $q := e^{2\pi i z}$ we had

$$\begin{aligned} E_4(z) &= 1 + 240 \sum_n^\infty \sigma_3(n) q^n, \quad E_6(z) = 1 - 504 \sum_n^\infty \sigma_5(n) q^n \\ E_8(z) &= 1 + 480 \sum_n^\infty \sigma_7(n) q^n, \quad E_{10}(z) = 1 - 264 \sum_n^\infty \sigma_9(n) q^n \\ E_{14}(z) &= 1 - 24 \sum_n^\infty \sigma_{13}(n) q^n, \end{aligned}$$

where

$$\sigma_k(n) := \sum_{\substack{m|n \\ m>0}} m^k.$$

By Claim 1.6 we have $E_8 = E_4^2$, thus

$$\begin{aligned} 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n &= 2 \cdot 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n + (240)^2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sigma_3(n)\sigma_3(m) q^{n+m} \\ \Leftrightarrow \sum_{n=1}^{\infty} \sigma_7(n) q^n &= \sum_{n=1}^{\infty} \sigma_3(n) q^n + 120 \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} \sigma_3(i)\sigma_3(n-i) q^n. \end{aligned}$$

Comparing coefficients we get for all $n \geq 1$

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{i=1}^{n-1} \sigma_3(i)\sigma_3(n-i).$$

Doing the exact same thing for the other equations in Claim 1.6 we get

$$\begin{aligned} 11\sigma_9(n) &= -10\sigma_3(n) + 21\sigma_5(n) + 5040 \sum_{i=1}^{n-1} \sigma_3(i)\sigma_5(n-i) \\ \sigma_{13}(n) &= 21\sigma_5(n) + 20\sigma_7(n) + 10080 \sum_{i=1}^{n-1} \sigma_5(i)\sigma_7(n-i). \end{aligned}$$

This is a result in number theory which we have derived via complex analysis.

2 Structure of the Graded Ring Generated by Modular Forms

From now on, we will identify $\mathbb{H}/\bar{\Gamma}$ and $\{z \in \mathbb{C} : |z| > 1, -1/2 < Re(z) < 1/2\} \cup \{-1/2 + ai \in \mathbb{C} : a \geq \sqrt{3}/2\} \cup \{e^{i\theta} \in \mathbb{C} : \pi/2 \leq \theta \leq 2\pi/3\}$.

Theorem 2.1. *For any k, M_{2k} can be generated by all monomials $G_2^{\alpha} G_3^{\beta}$, α, β integers ≥ 0 and $4\alpha + 6\beta = 2k$.*

Proof. We claim that for a fixed pair of α, β, s.t. $\alpha, \beta \geq 0, 4\alpha + 6\beta = 2k \geq 12$,

$$M_{2k} = S_{2k} \oplus \mathbb{C} G_4^{\alpha} G_6^{\beta}.$$

This is beacuse the right part of the above equation is a subspace of the left part and they have the same dimension as $\mathbb{C}$-vector spaces.
So we just need to show that S_{2k} can be generated by these monomials. Then we have $S_{2k} = \Delta M_{2k-12}$. Since $\Delta \in \mathbb{C} G_4^3 \oplus \mathbb{C} G_6^2$, we just need to show that M_{2k-12} can be generated by all monomials $G_4^{\alpha} G_6^{\beta}$ with α, β integers ≥ 0 and $4\alpha + 6\beta = 2k - 12$.
If we show that M_{2k} has this property, $\forall k < 6$, we will be done by induction.
For $k < 0$ and $k = 1$, $M_{2k} = \{0\}$ and there is no nonnegative solution for $4\alpha + 6\beta = 2k$.
For $k = 0, 4, 6, 8, 10$

$$\begin{aligned}
M_0 &= \mathbb{C} & 4\alpha + 6\beta = 0, \quad \alpha, \beta \geq 0 &\implies (\alpha, \beta) = (0, 0) \\
M_4 &= \mathbb{C} G_4 & 4\alpha + 6\beta = 4, \quad \alpha, \beta \geq 0 &\implies (\alpha, \beta) = (1, 0) \\
M_6 &= \mathbb{C} G_6 & 4\alpha + 6\beta = 6, \quad \alpha, \beta \geq 0 &\implies (\alpha, \beta) = (0, 1) \\
M_8 &= \mathbb{C} G_4^2 & 4\alpha + 6\beta = 8, \quad \alpha, \beta \geq 0 &\implies (\alpha, \beta) = (2, 0) \\
M_{10} &= \mathbb{C} G_4 G_6 & 4\alpha + 6\beta = 10, \quad \alpha, \beta \geq 0 &\implies (\alpha, \beta) = (1, 1)
\end{aligned}$$

□

Theorem 2.2. *$\forall k \geq 0$, all the monomials $G_4^{\alpha} G_6^{\beta}$ s.t. $\alpha, \beta \geq 0$ $4\alpha + 6\beta = 2k$ are linearly independent in M_{2k}.*

Proof. Set

$$f = \sum_{4\alpha + 6\beta = 2k, \alpha, \beta \geq 0} a_{\alpha,\beta} G_4^{\alpha} G_6^{\beta}, \quad a_{\alpha,\beta} \in \mathbb{C}.$$

If $f = 0$, we need to prove $a_{\alpha,\beta} = 0$.
Since $G_4(\rho) = 0$, we have

$$f(\rho) = \sum_{6\beta = 2k} a_{0,\beta} G_6^{\beta}(\rho) = 0.$$

Since $G_6(\rho) \neq 0$, $a_{0,\beta} = 0$.
Moreover, since $G_6(i) = 0$

$$f(i) = \sum_{4\alpha = 2k} a_{\alpha,0} G_4^{\alpha}(i) = 0,$$

and since $G_4(i) \neq 0$, $a_{\alpha,0} = 0$.
So

$$f = \sum_{4\alpha + 6\beta = 2k, \quad \alpha, \beta > 0} a_{\alpha,\beta} G_4^{\alpha} G_6^{\beta}.$$

So,

$$f/G_4G_6 = \sum_{4\alpha+6\beta=2k-10, \quad \alpha,\beta\geq 0} a_{\alpha,\beta}G_4^{\alpha}G_6^{\beta} = 0.$$

So to prove the linear independece in M_{2k}, we only need to show it in M_{2k-10}. And the linear independence is obvious for M_{2k} when $k = 0, 1, 2, 3, 4$, because they are $\{0\}$ or 1-dimensional vector spaces. $\square$

Theorem 2.3. *Modular forms of different weights are linearly independent.*

Proof. If we can prove

$$\sum_{i=1}^{k} f_i(z) = 0 \quad \forall z \implies f_i(z) = 0 \quad \forall z \quad i = 1, 2...k,$$

where f_i is an arbitriry weight 2(i-1) modular form, then we're done. Let $F(z) = \sum_{i=1}^{k} f_i(z) = 0 \quad \forall z$.

For an arbitrary fixed $z \in \mathbb{H}$, we find a $g_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \in \Gamma = SL_2(\mathbb{Z}) \quad \forall i = 1, 2...k$, such that all $c_iz + d_i$ are distinct. For example, one may put $g_i = \begin{bmatrix} 1 & i \\ 1 & i+1 \end{bmatrix}$. Then

$$F(g_i.z) = \sum_{j=1}^{k} (c_iz + d_i)^{j-1} f_j(z) = 0 \quad i = 1, 2...k$$

Set a Vandermonde matrix $A = ((c_iz + d_i)^{j-1})_{ij}$, whose determinant $\prod_{1<i<j<k}(c_jz + d_j - c_iz - d_i)$ is not zero beacuse of the distinct $c_iz + d_i$. Set a column vector $f = (f_i(z))_i$. Then

$$Af = (F(g_i.z))_i = (0)_i,$$

since $det(A) \neq 0$, $f = (f_i(z))_i = (0)_i$.
Since z is arbitrary, $f_i(z) = 0 \quad \forall z, \quad \forall i.$ $\square$

Theorem 2.4. *G_4 and G_6 are algebraically independent.*

Proof. A polynominal F of G_4 and G_6 is a finite summation of monominals $G_4^{\alpha}G_6^{\beta}$. Since the summation of monominals of the same weight $2k$ is a weight 2k modular form. Then we can write

$$F = \sum_{k\in P} f_{2k},$$

where P is a finite set of $\mathbb{N}$, f_{2k} is a weight $2k$ modular form generated by monominals of the same weight $2k$. If $F = 0$, by Theorem 2.3, $f_{2k} = 0 \quad \forall k \in P$. By Theorem 2.2, such monominals are linearly independent, so $\forall k \in P$, f_k is a zero polynominal of G_4 and G_6. Thus F is a zero polynominal. $\square$

Corollary 2.5. *Define a map $T: \quad C[X, Y] \longrightarrow \oplus_k M_{2k}$, which is generated by $X \mapsto G_4$ and $Y \mapsto G_6$. T is an isomorphism.*

Proof. T is injective because of Theorem 2.4. T is surjective beacuse of Theorem 2.1. $\square$

3 Structure of the Field of Weight 0 Modular Functions

Now we will consider the weight 0 modular functions, recall that a modular function is meromorphic on $\mathbb{H}/\bar{\Gamma} \cup \{\infty\}$. As we implied in the title, the set of all wight 0 modular functions is closed under addition and multiplication and the inversion on both aspects, so it is a field.

Definition 3.1. We put

$$j = 1728 \times 60^2 G_4^3/\Delta,$$

which is called *the modular invariant*. We use the coefficient 1728×60^2 to make its residue 1 at ∞.

Properties 3.2. i) j is a weight 0 modular function.

ii) j only has a simple pole at ∞ and is holomorphic on $\mathbb{H}$.

iii) j is a bijection from $\mathbb{H}/\bar{\Gamma}$ to $\mathbb{C}$.

Proof. *i)* Because G_4^3 and Δ are both modular forms of weight 12.
ii) $j = 1728 \times 60^2 G_4^3/\Delta$ has a simple pole at ∞ because Δ has a simple zero at ∞.
iii) Since

$$1728 \times 60^2 G_4^3 - \lambda\Delta = 0 \Longleftrightarrow 1728 \times 60^2 G_4^3/\Delta = \lambda,$$

we will prove that $\forall \lambda \in \mathbb{C}$, $f(z) = 1728 \times 60^2 G_4^3 - \lambda\Delta$ has a unique zero in $\mathbb{H}/\bar{\Gamma}$. Consider the valence formula for the weight 12 modular form f. Obviously, it's a modular form so it has no poles. Also, it is not 0 at ∞, since Δ vanishes at ∞ but G_4 not. We set

$$n = v_\infty(f) + \sum_{p\in\mathbb{H}/G}^{*} v_p(f)$$

$$n' = v_i(f)$$

$$n'' = v_\rho(f)$$

The only nonnegative integer solutions of $n + n'/2 + n''/3 = 1$ are $(1,0,0)$, $(0,2,0)$ and $(0,0,3)$, which all imply that it has a unique zero in $\mathbb{H}/\bar{\Gamma} \cup \{\infty\}$. And since f is not 0 at ∞, it has a unique zero in $\mathbb{H}/\bar{\Gamma}$.

□

Theorem 3.3. *Let f be a meromorphic function on $\mathbb{H}$. The following properties are equivalent:*

i) f is a modular function of weight 0.

ii) f is a quotient of two modular forms of the same weight.

iii) f is a rational function of j.

Proof. *ii)* $\Longrightarrow$ *i)* is trivial, we will prove *i)* $\Longrightarrow$ *iii)* and *iii)* $\Longrightarrow$ *ii)*.
i) $\Longrightarrow$ *iii)* Since j is holomorphic in $\mathbb{H}/\bar{\Gamma}$ we can mutiply any weight 0 modular function f by several times of weight 0 modular function $j - j(z_0)$ to kill the poles of f in z_0, since f has only finitely many poles by Claim 1.1.

So we can assume f has no poles in $\mathbb{H}/\bar{\Gamma}$. However, f is a weight 0 modular function, so it is a constant or it only has a pole in ∞. Let $g = \Delta^n f$, for some nonnegative integer n. Then g is a modular form weight $12n$, which can be generated by $G_4^\alpha G_6^\beta$ s.t. $4\alpha + 6\beta = 12n, \alpha, \beta \geq 0$. Without loss of generality, we assume $g = G_4^\alpha G_6^\beta$, and $4\alpha + 6\beta = 12n$, so $3|\alpha$ and $2|\beta$, so

$$f = \frac{g}{\Delta^n} = \left(\frac{G_4^3}{\Delta}\right)^{\alpha/3} \left(\frac{G_6^2}{\Delta}\right)^{\beta/2},$$

which is a rational function of j, since

$$\frac{G_4^3}{\Delta} = \frac{j}{1728 \times 60^2}$$

and

$$\frac{G_6^2}{\Delta} = \frac{60}{1728 \times 27 \times 140^2} j - \frac{1}{27 \times 140^2}.$$

iii) $\Longrightarrow$ *ii)* Let $F(j) = f_1(j)/f_2(j)$ be a rational function of j, and f_1 and f_2 polynomials of j. f_1 and f_2 are obvious modular functions only having poles at ∞. So $f_1\Delta^n$ and $f_2\Delta^n$ are modular forms of weight $12n$ for big enough n. Then

$$F = f_1\Delta^n / f_2\Delta^n.$$

□

Since j is a bijection, we have a bijection $j^* : f \mapsto f \circ j$ from the field of rational functions $\mathbb{C}(x)$ to the field of weight 0 modular functions induced by j, which is well defined because $f \circ j$ is a rational function of j and by Theorem 3.3 is a modular function of weight 0. The inverse of j^*, $j^{*-1} : h \mapsto h \circ j^{-1}$ is also well defined, which is trivial by Theorem 3.3. We can see it more clearly by a commutative diagram below.

$$\begin{tikzcd}
\mathbb{H}/\bar{\Gamma} \arrow[rr, leftrightarrow, "j", "j^{-1}"'] \arrow[dr, "h = f \circ j"'] & & \mathbb{C} \arrow[dl, "f = f \circ j \circ j^{-1} = h \circ j^{-1}"] \\
& \mathbb{C} &
\end{tikzcd}$$

Also, there is a famous result that meromorphic functions on $S^2 = \mathbb{C} \cup \{\infty\}$ are exactly the rational functions. We should notice that a weight 0 modular function is not only defined on $\mathbb{H}/\bar{\Gamma}$, it is actually a meromorphic function on $\mathbb{H}/\bar{\Gamma} \cup \{\infty\}$. So we can rewrite this relation in a more complete way including the point ∞.

4 Modular Forms of Complex Lattices

A complex lattice L is an additive subgroup of the form $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ in $\mathbb{C}$, s.t. $\omega_1/\omega_2 \in \mathbb{H}$. Denote the set of all complex lattices by $\mathbb{L}$.

Definition 4.1. A function $F : \mathbb{L} \longrightarrow \mathbb{C}$ is called a modular function of weight 2k, if F satisfies

$$F(L_1) = \lambda^{-2k} F(L_2),$$

where $L_1 = \lambda L_2$.

Note that any lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \lambda(\mathbb{Z}z + \mathbb{Z}) = \lambda L_z$, with $z = \omega_1/\omega_2 \in \mathbb{H}, \lambda = \omega_2$. Then for a weight 2k modular function F on lattices, $F(L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \lambda^{-2k} F(L_z)$, F is associated to a function $f(z) := F(\mathbb{Z}z+\mathbb{Z}) = F(L_z)$. Since (ω_1, ω_2) and $(a\omega_1+b\omega_2, c\omega_1+d\omega_2)$ generate the same lattice with $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma = SL_2(\mathbb{Z})$, f then satisfies the property

$$f(z) = (cz+d)^{-2k} f(\frac{az+b}{cz+d}),$$

which relates modular functions of lattices to ordinary modular function.
We put

$$G_4(L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \sum_{m,n\in\mathbb{Z},(m,n)\neq(0,0)} (m\omega_1 + n\omega_2)^{-4} = \sum_{0\neq a\in L} a^{-4}$$

and

$$G_6(L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \sum_{m,n\in\mathbb{Z},(m,n)\neq(0,0)} (m\omega_1 + n\omega_2)^{-6} = \sum_{0\neq a\in L} a^{-6}.$$

Then we set $g_4 = 60G_4$ and $g_6 = 140G_6$ for both lattice functions and for associated modular forms on $\mathbb{H}$.
It is easy to see g_4 and g_6 are modular functions of lattices and that

$$g_{2k}(L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \lambda^{-2k} g_{2k}(z) \quad k = 2, 3$$

with $z = \omega_1/\omega_2, \lambda = \omega_2$, and also $\Delta = g_4^3 - 27g_6^2$ and $j = 1728g_4^3/\Delta$.

Theorem 4.2. *For $A, B \in \mathbb{C}$, such that $A^3 - 27B^2 \neq 0$, there is a unique lattice L, such that $g_4(L) = A, g_6(L) = B$.*

Proof. We consider the equation

$$A = g_4(L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \lambda^{-4} g_4(z)$$

$$B = g_6(L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \lambda^{-6} g_6(z),$$

with $z = \omega_1/\omega_2, \lambda = \omega_2$. At least A, B cannot be both 0. Without loss of generality, we first assume $A \neq 0$. Then we can write

$$27B^2/A^3 = 27g_6(L)^2/g_4(L)^3 = 27g_6(z)^2/g_4(z)^3 = 1 - 1728/j(z)$$

By the property of j, we know that $1 - 1728/j(z)$ is a bijection from $\mathbb{H}/\bar{\Gamma}$ to $(\mathbb{C}\backslash\{1\}) \cup \{\infty\}$. Since $27B^2/A^2 \in \mathbb{C}\backslash\{1\}$, we can always find a unique solution $z \in \mathbb{H}/\bar{\Gamma}$. For the case $A = 0$, we can choose z, the zero of j, to satisfy the equation. Anyway, we can always find a unique solution $z \in \mathbb{H}/\bar{\Gamma}$, which gives us the lattice L_z. As z is fixed, we choose the value of λ, s.t.

$$A = \lambda^{-4} g_4(z) = (\lambda^2)^{-2} g_4(z)$$

$$B = \lambda^{-6} g_6(z) = (\lambda^2)^{-3} g_6(z)$$

It's easy to see that we can always choose a unique λ^2 to satisfy the above equations, which gives two solutions λ and $-\lambda$. Then $L = \lambda L_z = -\lambda L_z$ is the unique lattice we are looking for.

$\square$

Note that $A^3 - 27B^2$ is the discriminant of the cubic function $f(x) = 4x^3 + Ax + B$. And we know that $f(x)$ has three distinct roots $\Longleftrightarrow A^3 - 27B^2 \neq 0$.
On the other hand, $\forall L \in \mathbb{L}$, $g_4^3(L) - 27g_6^2(L) = \lambda^{-12}\Delta(z) \neq 0$.
So, each $L \in \mathbb{L}$ gives a cubic equation $y^2 = 4x^3 + g_4(L)x + g_6(L)$. This defines an elliptic curve on $\mathbb{C}$.

References

[1] A Course in Arithmetic by J.-P. Serre (Springer 1973)

[2] Introduction to Elliptic Curves and Modular Forms by Neal Koblitz (Springer 1993)