# The Classical Theta Function and the Riemann Zeta Function

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## 1 Introduction

The goal of this talk is to give a proof of the following result:

*The Riemann zeta function can be analytically extended to the whole complex plane except for a simple pole at* 1.

Recall that the Riemann zeta function is defined to be

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \mathrm{Re}(s) > 1$$

(the result above extends this domain to $\mathbb{C} \setminus \{1\}$).

To this end we will make use of the classical Jacobi theta function, its Mellin transform, and the Gamma function. Our main reference for this talk is [6, ch. II, §4].

While the Riemann zeta function is interesting on its own for its many number-theoretical properties, what is of most relevance to us is that our proof will serve to motivate the study of the relation between modular forms and their associated $L$-series and $L$-functions.

The process which we shall describe starts with the Jacobi theta function:

$$\theta(s) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 s}, \quad \mathrm{Re}(s) > 0.$$

which can be thought of as a modular form of weight $\frac{1}{2}$ (albeit we did not define modular forms of non-integer weight). We will go into more detail on this later on how this makes sense.

We then apply the *Mellin transform* on the theta function to "obtain" the Riemann zeta function up to some scaling and correction terms. Crucially, using the Mellin transform we are able to use various properties of the theta function itself, in particular the modularity of the theta function (Proposition 3.3), to obtain the desired analytic continuation of the Riemann zeta function.

More generally, one can start with a modular form $f$, write it in terms of its Fourier series (in the presentation given by Etienne and Aurel, they showed that any weakly modular function has a Fourier expansion. See Corollary 1.8 of their notes):

$$f(s) = \sum_{n \in \mathbb{Z}} a_n q^n = \sum_{n=0}^{\infty} a_n q^n \quad \text{(since modular forms are holomorphic at } \infty)$$

where $q = e^{2\pi i s}$. We then define a Dirichlet series with coefficients taken from the Fourier series called the *L-series* (associated with the modular form):

$$L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

In our case, the Riemann zeta function is *almost* the $L$-series of the theta function, there is some scaling needed (you can try writing out the Fourier series of the theta function, it will have coefficients all zero at the non-square indices, and 2 otherwise).

It turns out that one can also obtain the $L$-series, up to some correction terms and scaling, via the Mellin transform of the modular form, just as in the case of the theta function. As such, what is most important to take away from this talk, is that **one can essentially reproduce the techniques as we shall describe in this talk with minor adjustments, to obtain the analytic continuation of the $L$-series associated to a modular form. This analytic continuation is called the *$L$-function* associated to the modular form.**

The $L$-function associated to a modular form exhibit many properties that are direct analogues with those of the Riemann zeta function and often posses a great deal of number-theoretical properties, just as the Riemann zeta function does. For example, just as the Riemann zeta function can be written as an Euler product (thus indicating some relation with the prime numbers):

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

the $L$-series of a modular form can be written as an Euler product under certain (most) conditions as well, see Theorem 25.24 in [7].

We can sum up what we have described by the following diagram:

$$\begin{tikzcd}
\text{modular form} \ni f \arrow[r] \arrow[rr, bend right, "\text{Mellin transform}"'] & \text{fourier series of } f \arrow[r, "\text{put coefficients into Dirichlet series}"] & L\text{-series} \arrow[r, "\text{analytic continuation}"] & L\text{-function} \arrow[d] \\
 & & & \text{other properties}
\end{tikzcd}$$

Now one interesting question people have been asking is whether or not we can "reverse the arrows". That is to say, whether we can start with some function that satisfy some common properties that $L$-series or $L$-functions have, and show that it comes from a modular form. This question has a lot of interesting intricacies and is still being studied in various contexts. One of the earliest results by Hecke in 1935 [5] give sufficient conditions for a Dirichlet series to be the Mellin transform of a modular form, and a generalization to "higher dimensional" modular forms was proven by Weil in 1967 [8]. These are termed *converse theorems*. Since then there have been many results for various particular cases and generalizations to automorphic forms, other types of more general $L$-functions, etc.

In particular, in its most general setting it is studied in what is called the Langlands Program. We are unfamiliar with the language used in it, so we cannot go into any detail. It suffices to say that in this setting, extreme generalizations of "modular forms", and "$L$-functions" are considered, and there is ongoing research into undestanding the corresponding relation between them. For those interested, see [3], or maybe ask Professor Imamoglu or Alessandro about it.

There is however one special case of this question (of "reversing the arrows") worth noting, to which is answered by the Modularity theorem. Originally conjectured in various forms by Taniyama, Shimura, and Weil, and proven by Wiles and Taylor (never forget about the contribution of Richard Taylor!), the theorem states the following (rather cryptic) result:

**Theorem 1.1** (Taylor-Wiles Modularity theorem)**.** *Every semistable elliptic curve $E/\mathbb{Q}$ is modular.*

(Note: We now have a stronger version that does not require seminstability, due to Conrad, Diamond, and Taylor, see [2])

Roughly speaking, the theorem considers an $L$-series $L_E(s)$ defined by an elliptic curve $E$, and the theorem asserts that this series must come from a modular form (via the correspondence of the above diagram of course). Fun fact: using the techniques presented in this talk, a direct Corollary of this theorem is that the $L$-series $L_E(s)$ of the elliptic curve has an analytic continuation to an $L$-function!

This theorem was the last piece needed for the complete proof of Fermat's Last theorem, evidently a result of the combined efforts together with many others such as Gerhard Frey, Jean-Pierre Serre, Ken Ribet, etc.

To sum it up, this correspondence we have described between modular forms and their $L$-series and/or $L$-function is actively studied by many and has many wide ranging implications and connections with other topics of mathematics; what we will be presenting today represents only the most primitive example of it. To reiterate, what is most important is to recognize that the techniques we present today can be reproduced on modular forms to obtain the analytic continuation of its $L$-series. That is, starting from a modular form, we can realize its associated $L$-series via Mellin Transform up to some correction and scaling, we can then analytically extend the $L$-series to an $L$-function.

By the way, there will also be a talk on $L$-functions in a couple of weeks, so more details on all this to be expected.

## 2 Preliminaries: The Gamma Function

We need some useful facts on the Gamma function, which will appear later when we apply the Mellin transform on the theta function.

**Definition 2.1.** The Gamma function is defined to be

$$\Gamma(s) \coloneqq \int_0^\infty e^{-t} t^s \frac{\mathrm{d}t}{t}, \quad \mathrm{Re}(s) > 0.$$

It can be checked that the integral convergences absolutely for $\mathrm{Re}(s) > 0$ and defines an analytic function. The idea is to separate the integral:

$$\Gamma(s) \coloneqq \int_0^\infty e^{-t} t^s \frac{\mathrm{d}t}{t} = \int_0^1 e^{-t} t^{s-1} \mathrm{d}t + \int_1^\infty e^{-t} t^{s-1} \mathrm{d}t$$

The first term has a bound since the decreasing function $e^{-t}$ attains its maximum on $[0, 1]$ at 1, so:

$$\int_0^1 e^{-t} t^{s-1} \mathrm{d}t < \int_0^1 t^{s-1} \mathrm{d}t = \frac{1}{s}$$

For the second term, we use the fact that the exponential grows faster than any polynomial, therefore for any $s$ we can take $N \in \mathbb{N}$ large enough such that for any $t \geq N$ implies $e^{t/2} > t^{s-1}$. Therefore

$$\begin{aligned}
\int_1^\infty e^{-t} t^{s-1} \mathrm{d}t &= \int_1^N e^{-t} t^{s-1} \mathrm{d}t + \int_N^\infty e^{-t} t^{s-1} \mathrm{d}t \\
&< \int_1^N e^{-t} t^{s-1} \mathrm{d}t + \int_N^\infty e^{-t} e^{t/2} \mathrm{d}t \\
&= \int_1^N e^{-t} t^{s-1} \mathrm{d}t + \int_N^\infty e^{-t/2} \mathrm{d}t \\
&< \infty
\end{aligned}$$

See [1, p. 198] for complete details.

**Proposition 2.2.** *The Gamma function satisfies the functional equation*

$$\Gamma(s + 1) = s\Gamma(s).$$

*Proof.* We prove this directly using the definition and integration by parts:

$$\begin{aligned}
\Gamma(s+1) &= \int_0^\infty e^{-t} t^{s+1} \frac{\mathrm{d}t}{t} \\
&= [-e^{-t} t^s]_0^\infty + \int_0^\infty e^{-t} s t^s \frac{\mathrm{d}t}{t} \quad \text{(integration by parts)} \\
&= \lim_{t \to \infty} (-e^{-t} t^s) - 0 + s \int_0^\infty e^{-t} t^s \frac{\mathrm{d}t}{t} \\
&= s\Gamma(s).
\end{aligned}$$

□

**Corollary 2.3.** *The Gamma function has an analytic continuation on $\mathbb{C}$ with simple poles at $s = 0, -1, -2, \ldots$.*

*Proof.* The functional equation satisfied by the Gamma function in Proposition 2.2 can be used to inductively obtain values of the Gamma function on positive integers. First, re-writing the functional equation we get:

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}$$

So for any value of $s$ where $-1 < \mathrm{Re}(s) < 0$, and $s \neq 0$, $\Gamma(s)$ is well defined and analytic. Thus we have analytically extended $\Gamma(s)$ to include the strip $-1 < \mathrm{Re}(s) < 0$. Applying the functional equation as many times as needed, each time extending the function by another strip towards the negative reals, while adding a simple pole at the integers, we get at the $n$-th application of the functional equation:

$$\Gamma(s) = \frac{\Gamma(s+n+1)}{s(s+1)\ldots(s+n)}.$$

The right hand side defines an analytic function for all $s$ with $\mathrm{Re}(s) > -(n+1)$ and $s \neq 0, -1, -2, \ldots, -n$. Since $s+n+1$ lies in the right half plane for sufficiently large $n$, we see that $\Gamma(s)$ has an analytic continuation to the whole complex planes with simple poles at $0, -1, -2, \ldots$. Furthermore, by uniqueness of analytic continuation, it follows that the functional equation of the Gamma function holds for all $s \in \mathbb{C}$.

□

Another property of the Gamma function we will use is that its reciprocal $1/\Gamma(s)$ is entire. This can be derived from the next proposition. We will however omit the proof of the proposition, since it is rather technical and involves another representation of the Gamma function as an infinite product:

$$\Gamma(s) = \frac{1}{s}\prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^s}{1+\frac{s}{n}}.$$

The details can be found in [1].

**Proposition 2.4.** *The Gamma function satisfies the functional equation*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

*Proof.* See [1, p. 199]. □

**Corollary 2.5.** *The reciprocal of the Gamma function is entire, i.e.*

$$\frac{1}{\Gamma(s)} = \frac{\sin(\pi s)}{\pi}\Gamma(1-s)$$

*is entire.*

*Proof.* The only poles of the right hand side are simple ones at $s = 1, 2, 3, \ldots$. However, the sine function has simple zeros at $\pi n$ for all $n \in \mathbb{N}$. Thus, the poles of the right hand side are removable and so $1/\Gamma(s)$ is entire. □

**Exercise 2.6.** Show that $\Gamma(1/2) = \sqrt{\pi}$. Hint: you may use that $\int_0^\infty e^{-x^2}\mathrm{d}x = \sqrt{\pi}/2$.

# 3 The Jacobi Theta Function

Now to the main protagonist of this talk.

**Definition 3.1.** The *Jacobi theta function*, also known as the *classical theta function* is the function on right half plane defined by

$$\theta(s) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 s}, \quad \mathrm{Re}(s) > 0.$$

For the remainder of our talk, this function will just be referred to by the name *theta function*.

**Proposition 3.2.** *The theta function is a holomorphic function on the right half plane.*

*Proof.* Let $C$ be a compact subset of the right half plane. We will show that $\theta(s)$ converges uniformly and absolutely on it. Since each summand of the series is holomorphic, this will imply that the theta function is indeed holomorphic. For this let $y_0 = \min_{x \in C} |\mathrm{Re}(x)|$. Then for sufficiently large $n_0$ we have for all $n \geq n_0$ that $n^2/y_0 \geq n$. This implies for $x \in C$

$$\left| \sum_{|n| \geq n_0} e^{-\pi n^2 x} \right| \leq \sum_{|n| \geq n_0} e^{-\pi n^2 y_0} \leq \sum_{|n| \geq n_0} e^{-\pi n}.$$

Since the latter is the tail of a geometric series independent of $x$, we get absolute and uniform convergence of the theta function on $C$. $\square$

Our goal of this section is to show that the theta function satisfies a functional equation similar to the weakly modular condition of a modular form.

**Proposition 3.3.** *The theta function satisfies the functional equation*

$$\theta(s) = \frac{1}{\sqrt{s}} \theta\left(\frac{1}{s}\right), \quad \mathrm{Re}(s) > 0$$

Before we can provide a proof, we need some facts from Fourier analysis.

## The Fourier Transform

**Definition 3.4.** Let $\mathscr{S}$ be the vector space of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{C}$ which decrease at infinity faster than any negative power function, i.e.

$$\mathscr{S} \coloneqq \{f : \mathbb{R} \to \mathbb{C} : f \text{ is } C^\infty \text{ and } \lim_{x \to \pm\infty} |x|^n f(x) = 0 \quad \forall n \in \mathbb{N}\}.$$

This space is called *Schwartz space.*

Then for any $f \in \mathscr{S}$ we define the *Fourier transform of* $f$ to be the function

$$\hat{f}(y) \coloneqq \int_{-\infty}^{\infty} e^{-2\pi i x y} f(x) \mathrm{d}x. \tag{1}$$

**Exercise 3.5** (trivial)**.** Check that $\mathscr{S}$ is indeed a vector space.

**Proposition 3.6.** *The integral* (1) *converges for all* $y \in \mathbb{R}$ *and* $f \in \mathscr{S}$*.*

*Proof.* Since $f$ lies in the Schwartz space, there exists a constant $C > 0$ such that $|f(x)| \leq C|x|^{-2}$ for $|x| > 0$. This implies:

$$\left|\hat{f}(y)\right| \leq \int_{-\infty}^{\infty} \left|e^{-2\pi i x y} f(x)\right| \mathrm{d}x \leq \int_{-1}^{1} |f(x)| \mathrm{d}x + 2\int_{1}^{\infty} C|x|^{-2}.$$

Clearly both integrals exist and so the Fourier transform converges. $\square$

**Lemma 3.7.** *If* $b > 0$ *and* $g(x) = f(bx)$*, then*

$$\hat{g}(y) = \frac{1}{b} \hat{f}\left(\frac{y}{b}\right).$$

*Proof.* We compute $\hat{g}(y)$ directly:

$$\begin{aligned}
\hat{g}(y) &= \int_{-\infty}^{\infty} e^{-2\pi ixy} f(bx)\mathrm{d}x \\
&= \int_{-\infty}^{\infty} e^{-2\pi i \frac{z}{b} y} f(z)\frac{\mathrm{d}z}{b} \quad \text{(make a change of variables } z = bx\text{)} \\
&= \frac{1}{b}\int_{-\infty}^{\infty} e^{-2\pi i \frac{y}{b} z} f(z)\mathrm{d}z \\
&= \frac{1}{b}\hat{f}\left(\frac{y}{b}\right).
\end{aligned}$$

$\square$

**Proposition 3.8.** *If $f(x) = e^{-\pi x^2}$, then $\hat{f} = f$.*

*Proof.* First note that

$$\hat{f}'(y) = \frac{\mathrm{d}}{\mathrm{d}y}\int_{-\infty}^{\infty} e^{-2\pi ixy} f(x)\mathrm{d}x = -2\pi i\int_{-\infty}^{\infty} e^{-2\pi ixy} x e^{-\pi x^2}\mathrm{d}x. \tag{2}$$

Differentiation under the integral is fine, since

$$\left|\frac{\mathrm{d}}{\mathrm{d}y}(e^{-2\pi ixy}e^{-\pi x^2})\right| \le \left|2\pi x e^{-\pi x^2}\right| \tag{3}$$

which is integrable.

Now we integrate the right-hand side of (2) by parts to get

$$\begin{aligned}
\hat{f}'(y) =\ & -2\pi i e^{-2\pi ixy}\frac{1}{-2\pi}e^{-\pi x^2}\Big|_{-\infty}^{\infty} + 2\pi i\int_{-\infty}^{\infty} -2\pi i y e^{-2\pi ixy}\frac{e^{-\pi x^2}}{-2\pi}\mathrm{d}x \\
&= -2\pi y\int_{-\infty}^{\infty} e^{-2\pi ixy} f(x)\mathrm{d}x \\
&= -2\pi y\hat{f}(y)
\end{aligned}$$

Therefore $\hat{f}$ satisfies the differential equation

$$\hat{f}'(y) = -2\pi y\hat{f}(y),$$

which has solution

$$\hat{f}(y) = Ce^{-\pi y^2}.$$

We can solve for $C$ by setting $y = 0$:

$$C = \hat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi x^2}\mathrm{d}x = 1.$$

Therefore we conclude

$$\hat{f}(y) = e^{-\pi y^2},$$

as desired. $\square$

We describe such a function with the property shown in Proposition 3.8 as being *self-dual*.

**Remark.** We are justified to pass the differential under the integral in the beginning of the above proof (equation (2)) by the following fact from abstract integration theory, the (general) Leibniz integral rule:

Let $X$ be an open subset of $\mathbb{R}$ , and $\Omega$ be a measure space. Suppose $f : X \times \Omega \to \mathbb{R}$ satisfies the following conditions:

1. $f(x,\omega)$ is a Lebesgue-integrable function of $\omega$ for each $x \in X$.

2. For almost all $\omega \in \Omega$, the derivative $f_x$ exists for all $x \in X$.

3. There is an integrable function $\theta : \Omega \to \mathbb{R}$ such that $|f_x(x,\omega)| \leq \theta(\omega)$ for all $x \in X$ and almost every $\omega \in \Omega$.

Then for all $x \in X$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\Omega} f(x,\omega)\mathrm{d}\omega = \int_{\Omega} f_x(x,\omega)\mathrm{d}\omega.$$

The proof of which chiefly uses the dominated convergence theorem. See [4, p. 125] for details.

**Proposition 3.9** (Poisson Summation Formula)**.** *If $f \in \mathscr{S}$, then*

$$\sum_{m\in\mathbb{Z}} g(m) = \sum_{m\in\mathbb{Z}} \hat{g}(m).$$

*Proof.* Define

$$G(x) = \sum_{n\in\mathbb{Z}} g(x+n).$$

Then $G(x)$ is 1-periodic and has Fourier series

$$G(x) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi ikx}, \tag{4}$$

where we can calculate the Fourier coefficient

$$\begin{aligned} c_k &= \int_0^1 \sum_{n\in\mathbb{Z}} g(x+n)e^{-2\pi ikx}\mathrm{d}x \\ &= \sum_{n\in\mathbb{Z}} \int_0^1 g(x+n)e^{-2\pi ikx}\mathrm{d}x \\ &= \sum_{n\in\mathbb{Z}} \int_n^{n+1} g(x)e^{-2\pi ikx}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} g(x)e^{-2\pi ikx}\mathrm{d}x \\ &= \hat{g}(k). \end{aligned}$$

The second equality is justified, because $g \in \mathscr{S}$ implies there exist constants $C_m > 0$ s.t. $|g(x)| \leq C_m|x|^{-m}$ for all $m \in \mathbb{N}$. Hence for sufficiently large $n_0$

$$\sum_{|n|\geq n_0} |g(x+n)| \leq \sum_{|n|\geq n_0} C_m|x+n|^{-m}$$

for all $m \in \mathbb{N}$ and thus the tail of the series gets arbitrarily small. This implies $G(x)$ converges absolutely and uniformely on all compact sets.

Therefore we can write (4) as

$$G(x) = \sum_{k\in\mathbb{Z}} \hat{g}(k)e^{2\pi ikx}.$$

In particular setting $x = 0$ we get

$$G(0) = \sum_{n\in\mathbb{Z}} g(n) = \sum_{k\in\mathbb{Z}} \hat{g}(k)$$

as desired. ∎

We are now ready to prove the functional equation of the theta function:

*Proof of Proposition 3.3.* Recall that we want to show that

$$\theta(s) = \frac{1}{\sqrt{s}}\theta(1/s)$$

for all $s$ in the right half plane. We do this by showing that both sides agree on the positive real axis. Then, since both sides of the equation define an holomorphic function on the right half plane which are identical on the positive real axis, we are done by uniqueness of analytic continuation. Note also that we have to choose the branch of the square root with positive real part to make the right hand side well defined.

So fix $t > 0$ and define

$$g(x) = e^{-\pi t x^2}.$$

We can write this function as

$$g(x) = f(\sqrt{t}x), \quad \text{where } f(x) = e^{-\pi x^2}.$$

Then by Lemma 3.7, we have

$$\hat{g}(y) = \frac{1}{\sqrt{t}}\hat{f}\left(\frac{y}{\sqrt{t}}\right)$$

and by Proposition 3.8,

$$\hat{g}(y) = \frac{1}{\sqrt{t}}f\left(\frac{y}{\sqrt{t}}\right) = \frac{1}{\sqrt{t}}e^{-\pi y^2/t}.$$

Now by the Poisson summation formula (Proposition 3.9), we have

$$\sum_{m\in\mathbb{Z}} e^{-\pi t m^2} = \sum_{m\in\mathbb{Z}} \frac{1}{\sqrt{t}}e^{-\pi m^2/t},$$

which we can re-write as

$$\theta(t) = \frac{1}{\sqrt{t}}\theta\left(\frac{1}{t}\right).$$

$\square$

**Remark.** Note that we required a modular form $f$ of weight $k$ to be a 1-periodic meromomorphic function on the upper half plane satisfying $f(-1/\tau) = \tau^k f(\tau)$ for all $\tau \in \mathbb{H}$. Now in that context, the theta function can be understood as a modular form of "half integer weight". Although we did not define such modular forms, the functional equation of the theta function gives a hint on what is meant. Furthermore, if you want to be precise, we have to make a transformation to change the domain of definition to the upper half plane and normalize the theta function to make it 1-periodic.

Next, we will study the behaviour of the theta functions near zero and infinity.

**Proposition 3.10.** *There exists a constant $C > 0$ such that for all sufficiently small $t > 0$ the following inequality holds:*

$$\left|\theta(t) - t^{-1/2}\right| < e^{-C/t}.$$

*Proof.* Note that we can rewrite the theta function as

$$\theta(t) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 t}. \tag{5}$$

Using this fact and the functional equation for the theta function, we get

$$\left|\theta(t) - t^{-1/2}\right| = \left|t^{-1/2}(\theta(1/t) - 1)\right| = t^{-1/2}2\sum_{n=1}^{\infty} e^{-\pi n^2/t}.$$

Now suppose $t > 0$ is sufficiently small so that

$$4e^{-1/t} < \sqrt{t} \quad \text{and} \quad e^{-3\pi/t} < 1/2.$$

Then we get

$$\begin{aligned}
\left|\theta(t) - t^{-1/2}\right| &= t^{-1/2} 2 \sum_{n=1}^{\infty} e^{-\pi n^2/t} \\
&< \frac{1}{2} e^{1/t} \sum_{n=1}^{\infty} e^{-\pi n^2/t} \\
&= \frac{1}{2} e^{-(\pi-1)/t} \sum_{n=1}^{\infty} e^{-\pi(n^2-1)/t} \\
&= \frac{1}{2} e^{-(\pi-1)/t} \sum_{n=1}^{\infty} e^{-\pi(n+1)(n-1)/t} \\
&\leq \frac{1}{2} e^{-(\pi-1)/t} \sum_{n=1}^{\infty} e^{-3\pi(n-1)/t} \\
&\leq \frac{1}{2} e^{-(\pi-1)/t} \sum_{n=1}^{\infty} 2^{-(n-1)} \\
&= e^{-(\pi-1)/t}.
\end{aligned}$$

Thus, $C = \pi - 1$ does the job. □

**Corollary 3.11.** *The limit of $\theta(t)$ as $t$ goes to infinity is $1$ and for sufficiently large $t > 0$, there exists a positive constant $C$ such that*

$$|\theta(t) - 1| \leq e^{-Ct}/\sqrt{t}.$$

*Proof.* If $t$ is large, then $1/t$ is small, so the previous proposition tells us that

$$\left|\theta(1/t) - \sqrt{t}\right| \leq e^{-Ct}.$$

Applying the functional equation we get

$$\left|\sqrt{t}\theta(t) - \sqrt{t}\right| \leq e^{-Ct}.$$

□

## The Mellin Transform

The Gamma function is a special case of a construction known as the *Mellin Transform*:

**Definition 3.12.** Given a function $f(t)$ defined on the positive real axis, its *Mellin transform* is the function $\mathcal{M}_f(s)$ defined to be

$$\mathcal{M}_f(s) := \int_0^{\infty} f(t) t^s \frac{\mathrm{d}t}{t},$$

which is defined for all values $s$ for which the integral exists.

Therefore, $\Gamma(s)$ is the Mellin transform of $e^{-t}$.

## 4 The Riemann Zeta Function

**Theorem 4.1.** *The Riemann zeta function $\zeta(s)$ for $\mathrm{Re}(s) > 1$ extends analytically onto the whole complex plane, except for a simple pole at $s = 1$ with residue 1. Furthermore, let*

$$\Lambda(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

*then*

$$\Lambda(s) = \Lambda(1-s),$$

*i.e. $\zeta(s)$ satisfies the functional equation*

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

*Proof.* The basic idea behind the proof is to consider the Mellin transform $\int_0^\infty t^s\theta(t)\frac{\mathrm{d}t}{t}$ of the theta function. However, the preceding proposition and corollary tell us that that the theta function behaves like $t^{-1/2}$ at 0 and converges rapidly to 1 at infinity. So we have to introduce some correction terms to make the integral converge. In addition, we have to replace $s$ by $s/2$ as otherwise we would get $\zeta(2s)$. So define

$$\phi(s) := \int_1^\infty (\theta(t)-1)t^{s/2}\frac{\mathrm{d}t}{t} + \int_0^1 \left(\theta(t) - \frac{1}{\sqrt{t}}\right)t^{s/2}\frac{\mathrm{d}t}{t}.$$

Now Proposition 3.10 implies that the theta function converges rapidly to $1/\sqrt{t}$ as $t$ goes to zero, so the second integral converges. Similarly, Corollary 3.11 implies that the first integral converges. Evaluating the second integral, assuming $\mathrm{Re}(s) > 1$, we get:

$$\begin{aligned}
\int_0^1 \left(\theta(t) - \frac{1}{\sqrt{t}}\right)t^{s/2}\frac{\mathrm{d}t}{t} &= \int_0^1 \theta(t)t^{s/2}\frac{\mathrm{d}t}{t} - \int_0^1 t^{(s-1)/2}\frac{\mathrm{d}t}{t} \\
&= \int_0^1 \left(\sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}\right)t^{s/2}\frac{\mathrm{d}t}{t} - \frac{2}{s-1} \\
&\overset{(5)}{=} \int_0^1 t^{s/2}\frac{\mathrm{d}t}{t} + 2\int_0^1 \left(\sum_{n=1}^{\infty} e^{-\pi n^2 t}\right)t^{s/2}\frac{\mathrm{d}t}{t} + \frac{2}{1-s} \\
&= 2\sum_{n=1}^{\infty}\int_0^1 e^{-\pi n^2 t}t^{s/2}\frac{\mathrm{d}t}{t} + \frac{2}{s} + \frac{2}{1-s}.
\end{aligned}$$

Therefore for $\mathrm{Re}(s) > 1$ we have

$$\begin{aligned}
\phi(s) &= 2\sum_{n=1}^{\infty}\int_1^\infty e^{-\pi n^2 t}t^{s/2}\frac{\mathrm{d}t}{t} + 2\sum_{n=1}^{\infty}\int_0^1 e^{-\pi n^2 t}t^{s/2}\frac{\mathrm{d}t}{t} + \frac{2}{s} + \frac{2}{1-s} \\
&= 2\sum_{n=1}^{\infty}\int_0^\infty e^{-\pi n^2 t}t^{s/2}\frac{\mathrm{d}t}{t} + \frac{2}{s} + \frac{2}{1-s}.
\end{aligned}$$

After the substitution $t \mapsto 1/t$ we get

$$\begin{aligned}
\frac{1}{2}\phi(s) &= \sum_{n=1}^{\infty}(\pi n^2)^{-s/2}\Gamma\left(\frac{s}{2}\right) + \frac{1}{s} + \frac{1}{1-s} \\
&= \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) + \frac{1}{s} + \frac{1}{1-s}.
\end{aligned}$$

Now since both integrals in the definition of $\phi(s)$ converge, it is in fact an entire function. So we can define an analytic continuation of the zeta function on the whole complex plane, which agrees with $\zeta(s)$ for $\mathrm{Re}(s) > 1$, by

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)}\left(\frac{1}{2}\phi(s) - \frac{1}{s} - \frac{1}{1-s}\right),$$

whose only possible poles are at $s = 0$ or $s = 1$ as $\phi(s)$ and $1/\Gamma(s)$ are entire. However, we can replace $s\Gamma(s/2)$ in the denominator by $2/(s/2)\Gamma(s/2) = 2\Gamma(s/2+1)$ which converges to $2\Gamma(1)$ as $s$ goes to zero. Hence it is nonzero and the only pole is at 1. We now compute the residue there using Exercise 2.6:

$$\lim_{s\to 1}(s-1)\frac{\pi^{s/2}}{\Gamma(s/2)}\left(\frac{1}{2}\phi(s) - \frac{1}{s} - \frac{1}{1-s}\right) = \frac{\pi^{1/2}}{\Gamma(1/2)} = 1.$$

We are therefore left with showing that $\Lambda(s) = \Lambda(1-s)$. Since $\Lambda(s) = \frac{1}{2}\phi(s) - 1/s - 1/(1-s)$ and the term $-1/s - 1/(1-s)$ does not change if we replace $s$ by $1-s$, it suffices to check that $\phi(s) = \phi(1-s)$:

$$\begin{aligned}
\phi(s) &= \int_1^\infty t^{s/2}(\theta(t)-1)\frac{\mathrm{d}t}{t} + \int_0^1 t^{s/2}\left(\theta(t) - 1/\sqrt{t}\right)\frac{\mathrm{d}t}{t} \\
&= \int_0^1 y^{-s/2}(\theta(1/y)-1)\frac{\mathrm{d}y}{y} + \int_1^\infty y^{-s/2}(\theta(1/y) - \sqrt{y})\frac{\mathrm{d}y}{y} \\
&= \int_0^1 y^{-s/2}(\sqrt{y}\theta(y)-1)\frac{\mathrm{d}y}{y} + \int_1^\infty y^{-s/2}(\sqrt{y}\theta(y) - \sqrt{y})\frac{\mathrm{d}y}{y} \\
&= \int_0^1 y^{(1-s)/2}(\theta(y)-1/\sqrt{y})\frac{\mathrm{d}y}{y} + \int_1^\infty y^{(1-s)/2}(\theta(y) - \sqrt{y})\frac{\mathrm{d}y}{y} \\
&= \phi(1-s),
\end{aligned}$$

where we used the transformation $y = 1/t$ in the second equality and the transformation law for the theta function in the third. Thus, the proof is complete. □

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