The Petersson Inner Product and Poincaré Series

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1 The Petersson Inner Product

Remark. In the second talk, we learned that $\mathcal{M}_k(\Gamma = SL_2(\mathbb{Z})) = 0$ if k is odd. Therefore, in the following content, k is even when we mention the *weight* of modular forms and related concepts, unless otherwise mentioned.

Definition 1.1. *On the upper half plane $\mathbb{H}$, we define the hyperbolic measure*

$$d\mu(z) = \frac{dxdy}{y^2}, \quad z = x + iy \in \mathbb{H}.$$

Proposition 1.1. *The hyperbolic measure $d\mu$ is invariant under $GL_2^+(\mathbb{R})$ of $\mathbb{H}$, i.e. $d\mu(\alpha(z)) = d\mu(z)$ for all $\alpha \in GL_2^+(\mathbb{R})$, $z \in \mathbb{H}$. Hence, $d\mu$ is also invariant under $\Gamma = SL_2(\mathbb{Z})$.*

Proof. Let A $\subseteq \mathbb{H}$ be a Borel set, $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and let $T(z) = \frac{az+d}{cz+d}$ for every $z \in \mathbb{H}$. Write $z = x + iy \in \mathbb{H}$. One can check easily that

$$Im(T(z)) = \frac{y}{|cz+d|^2}, \quad T'(z) = \frac{1}{(cz+d)^2}$$

Now, we check the invariance:

$$\mu(T(A)) = \int_{T(A)} \frac{dudv}{v^2} = \int_A |T'(z)|^2 \frac{|cz+d|^4}{y^2} dxdy = \int_A \frac{|cz+d|^4}{y^2} \frac{dxdy}{|cz+d|^4} = \int_A \frac{dxdy}{y^2} = \mu(A).$$

□

Definition 1.2. *Let $f,\ g \in \mathcal{S}_k$, where $\mathcal{S}_k$ denotes the space of cusp forms of weight k. We define the Petersson Inner Product as the following:*

$$\langle \cdot, \cdot \rangle : \mathcal{S}_k \times \mathcal{S}_k \to \mathbb{C}, \quad (f, g) \mapsto \int_{\Gamma \backslash H} f(z)\overline{g(z)}(Im(z))^k d\mu(z).$$

Lemma 1.1. *If f and g are modular forms of weight k, then the function $f(z)\overline{g(z)}(Im(z))^k$ is $SL_2(\mathbb{Z})$-invariant.*

Proof. Fix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, and recall that if f is a modular form then $f(\gamma z) = (cz+d)^k f(z)$. Thus,

$$f(\gamma z)\overline{g(\gamma z)}Im(\gamma z)^k = (cz+d)^k f(z)\overline{(cz+d)^k g(z)}\tfrac{y^k}{|cz+d|^{2k}} =$$
$$|cz+d|^{2k} f(z)\overline{g(z)}\tfrac{y^k}{|cz+d|^{2k}} = f(z)\overline{g(z)}Im(z)^k,$$

as desired. □

Proposition 1.2. *The Petersson inner product is well-defined and Hermitian.*

Proof. Let $f, g, h \in \mathcal{S}_k$, and $c \in \mathbb{C}$. Since f and g are cusp forms, they decay exponentially and hence the integral converges. The integral is also independent of the choice of fundamental domains because $d\mu$ and $f(z)\overline{g(z)}(Im(z))^k$ are $SL_2(\mathbb{Z})$-invariant, as we showed above. Now we verify it is Hermitian.

$$\langle f+g, h\rangle = \int_{\Gamma\backslash H}(f(z)+g(z))\overline{h(z)}(Im(z))^k d\mu(z) =$$
$$\int_{\Gamma\backslash H} f(z)\overline{h(z)}(Im(z))^k d\mu(z) + \int_{\Gamma\backslash H} g(z)\overline{h(z)}(Im(z))^k d\mu(z) = \langle f, h\rangle + \langle g, h\rangle$$

$$\langle cf, g\rangle = \int_{\Gamma\backslash H} cf(z)\overline{g(z)}(Im(z))^k d\mu(z) =$$
$$\text{c}\int_{\Gamma\backslash H} f(z)\overline{g(z)}(Im(z))^k d\mu(z) = c\langle f, g\rangle.$$

$$\overline{\langle g, f\rangle} = \overline{\int_{\Gamma\backslash H} g(z)\overline{f(z)}(Im(z))^k d\mu(z)} = \int_{\Gamma\backslash H} f(z)\overline{g(z)}(Im(z))^k \overline{d\mu(z)} =$$
$$\langle f, g\rangle.$$

$$\text{Assume } f \neq 0, \text{ then } \langle f, f\rangle = \int_{\Gamma\backslash H} |f(z)|^2 (Im(z))^k d\mu(z) > 0.$$

□

Lemma 1.2. *A finite-dimensional inner product space V over $\mathbb{C}$ is a Hilbert space.*

Proof. Clearly, $V \cong \mathbb{C}^n$. Let $dim(V) = n$, and $\{v_1, v_2, ..., v_n\}$ an orthonormal basis of V, and $\{e_1, e_2, ..., e_n\}$ an orthonormal basis of $\mathbb{C}^n$. Define a linear isomorphism $T : V \to \mathbb{C}^n, T(v_i) = e_i$ for $i = 1, ..., n$. Clearly, $\langle T(v_i), T(v_j)\rangle = \langle e_i, e_j\rangle = \delta_{ij} = \langle v_i, v_j\rangle$, so T also preserves the inner product structure, hence V is also a Hilbert space. □

Theorem 1.3. *The space of cusp forms of weight k, $S_k(\Gamma)$, is a Hilbert space.*

Proof. By Corollary 1.5 in the third talk, $\mathcal{M}_k$ is finite-dimensional, so is $\mathcal{S}_k$. Hence, Lemma 1.1 shows that $\mathcal{S}_k$ is indeed a Hilbert space. □

Recall the Riesz Representation Theorem from functional analysis:

Theorem 1.4. *Let V be a finite-dimensional Hilbert space over $\mathbb{C}$ with an Hermitian inner product $\langle , \rangle$, and ϕ a linear functional $V \to \mathbb{C}$. For each $v \in V$, there exists a unique w such that $\langle v, w\rangle = \phi(v)$.*

Proof. [1] □

Recall that a cusp form f has a Fourier expansion $f(z) = \sum_{n>0} a_n e^{2\pi i n z}$. Hence, we can define a linear functional

$$\phi_m^k : \mathcal{S}_k \to \mathbb{C}, \quad f \mapsto a_m.$$

Let $f(z) = \sum_{n>0} a_n e^{2\pi i n z}$ be an arbitrary cusp form of weight k. By Theorem 1.4, we know that there exists a unique cusp form of weight k, here denoted as P_m^k, such that

$$\langle f, P_m^k\rangle = \phi_m^k(f) = a_m.$$

Our next goal is to construct such a type of cusp forms, namely Poincaré series.

2 The General Construction of Poincaré Series

Definition 2.1. *A function $\mu : \Gamma \times \mathbb{H} \to \mathbb{C}^\times$ is called an automorphy factor if for each $\alpha, \beta \in \Gamma, \mu_\alpha$ and μ_β are holomorphic on $\mathbb{H}$ and $\mu_{\alpha\beta}(z) = \mu_\alpha(\beta z)\mu_\beta(z)$.*

Example 2.1. $\mu_\gamma(z) = (cz+d)^k, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$

We try to construct functions f on $\mathbb{H}$ satisfying the automorphic condition

$$f(\gamma z) = \mu_\gamma(z) f(z) \quad \forall \gamma \in \Gamma,$$

for an arbitrary automorphy factor μ_γ.

The idea is attempting to write f as an average of a holomorphic function over Γ. Specifically, we define $f(z) = \sum_{\gamma\in\Gamma} \frac{h(\gamma z)}{\mu_\gamma(z)}$, and once we ensure that it is convergent, we can show that this function satisfies the desired properties.

Lemma 2.1. $f(z) = \sum_{\gamma\in\Gamma} \frac{h(\gamma z)}{\mu_\gamma(z)}$ *satisfies the automorphic condition.*

Proof. Choose an arbitrary $\alpha \in \Gamma$, we have:

$$f(\alpha z) = \sum_{\gamma\in\Gamma} \frac{h(\gamma\alpha z)}{\mu_\gamma(\alpha z)} = \sum_{\gamma\in\Gamma} \frac{h(\gamma\alpha z)\mu_\alpha(z)}{\mu_{\gamma\alpha}(z)} = \mu_\alpha(z) f(z),$$

since the bijection $\Gamma \to \Gamma\alpha, \gamma \mapsto \gamma\alpha$ is easily constructed. □

We need the series to converge absolutely, and uniformly on compact subsets in $\mathbb{H}$, but there are many $\gamma \in \Gamma$ such that $\mu_\gamma(z) \equiv 1$, so we define the collection of such elements as

$$\Gamma_\infty = \{\gamma \in \Gamma \mid \mu_\gamma \equiv 1\}.$$

Lemma 2.2. *Γ_∞ is a subgroup in Γ.*

Proof. Let $\alpha, \gamma \in \Gamma_\infty$, then for every $z \in \mathbb{H}$

$$\mu_{\alpha\gamma}(z) = \mu_\alpha(\gamma z)\mu_\gamma(z) = \mu_\alpha(\gamma z) = 1;$$

Let I be the identity matrix, we have:

$$\mu_{II}(z) = \mu_I(z) = \mu_I(z)\mu_I(z) \Rightarrow \mu_I(z) = 1;$$

Let γ^{-1} be the inverse of γ, we have:

$$\mu_{\gamma\gamma^{-1}}(z) = 1 = \mu_\gamma(\gamma^{-1}z)\mu_{\gamma^{-1}}(z) = \mu_{\gamma^{-1}}(z),$$

hence, $\gamma^{-1} \in \Gamma_\infty$. □

Denote $\mathcal{R} = \Gamma/\Gamma_\infty$, and let h be a Γ_∞-invariant function on $\mathbb{H}$. We change our definition of f above to the following:

$$f : \mathbb{H} \to \mathbb{C}, \quad f(z) = \sum_{\gamma\in\mathcal{R}} \frac{h(\gamma z)}{\mu_\gamma(z)}.$$

Lemma 2.3. *f is well-defined and still satisfies the automorphic condition.*

Proof. Suppose γ and γ' are in the same coset, then $\gamma = \beta\gamma'$ for some $\beta \in \Gamma_\infty$, then,

$$h(\gamma z) = h(\beta\gamma' z) = h(\gamma' z), \text{ and}$$
$$\mu_\gamma(z) = \mu_{\beta\gamma'}(z) = \mu_\beta(\gamma' z)\mu_{\gamma'}(z) = \mu_{\gamma'}(z).$$

Choose an arbitrary $\alpha \in \Gamma$, we have:

$$f(\alpha z) = \sum_{\gamma\in\mathcal{R}} \frac{h(\gamma\alpha z)}{\mu_\gamma(\alpha z)} = \sum_{\gamma\in\mathcal{R}} \frac{h(\gamma\alpha z)}{\mu_{\gamma\alpha}(z)}\mu_\alpha(z) = \sum_{\gamma\in\mathcal{R}} \frac{h(\gamma z)}{\mu_\gamma(z)}\mu_\alpha(z) = \mu_\alpha(z) f(z)$$

because both $\mathcal{R}$ and $\mathcal{R}\alpha$ are sets of representations of Γ/Γ_∞

□

Now let $\mu_\gamma(z) = (cz+d)^k$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

Lemma 2.4. $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, n \in \mathbb{Z}\}$, *hence*
$\mathcal{R} = \{\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), c \geq 0\} = \{(c,d) \in \mathbb{Z}^2 \mid c \geq 0, c \text{ and } d \text{ coprime}\}$

Proof. Clearly, $\{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, n \in \mathbb{Z}\} \subseteq \Gamma_\infty$. Conversely, suppose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty$. Then $\mu_\gamma(z) = d^k = 1$ for every $z \in \mathbb{H}$; since k is even, d can either be 1 or -1. c must be zero because otherwise $\mu_\gamma = (ci+1)^k$ or $(ci-1)^k$, and neither can be 1 for any $k > 0$. The value of b does not affect neither the values of μ_γ nor the determinant of γ, so b can be any integer. d must be equal to a, ensuring the determinant of γ to be 1.

Denote $\pm T^n = \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, n \in \mathbb{Z}$. Suppose $c \geq 0$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\alpha = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, and $\overline{\gamma} = \overline{\alpha} \in \mathcal{R}$, then $\gamma = \pm T^n \alpha$ for some $n \in \mathbb{Z}$. Then, apparently $c = c', d = d'$. Conversely, suppose $\alpha = \begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$. Observe that since $ad - bc = a'd - b'c = 1$, $\gcd(c,d) = 1$, and $d(a-a') = c(b-b')$, so $c \mid d(a-a')$, $d \mid c(b-b')$, and $\frac{a-a'}{c} = \frac{b-b'}{d}$. As we showed, $c \nmid d$, $d \nmid c$, so $c \mid (a-a')$, $d \mid (b-b')$. Observe that $T^n\alpha = \begin{pmatrix} a'+cn & b'+dn \\ c & d \end{pmatrix}$. Therefore, set $n = \frac{a-a'}{c} = \frac{b-b'}{d}$, we are done. If $c < 0$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -T^0 \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$.
Denote $\mathcal{R}' = \{(c,d) \in \mathbb{Z}^2 \mid c \geq 0, \text{c and d coprime}\}$. Define a function

$$f : \mathcal{R} \to \mathcal{R}', \quad \begin{pmatrix} * & * \\ c & d \end{pmatrix} \mapsto (c,d)$$

Clearly, f is well-defined. f is injective: if $f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = f(\begin{pmatrix} p & q \\ r & s \end{pmatrix}) = (c,d)$, then $r = c, s = d$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} p & q \\ c & d \end{pmatrix}$. f is surjective since for every pair of coprime integers c and d, there exists integers a and b such that $ad - bc = 1$. □

The equivalence of $\mathcal{R}$ and $\mathcal{R}'$ makes our definition of Poincaré series neater.

Definition 2.2. *Denote $e(z) = e^{2\pi iz}$, then the m^{th} Poincaré series of weight k for Γ is defined as:*

$$P_m^k(z) = \textstyle\sum_{\gamma\in\Gamma/\Gamma_\infty} j_\gamma(z)^{-k} e(m\gamma z) = \sum_{(c,d)\in\mathcal{R}} (cz+d)^{-k} e^{2\pi i m\gamma z}.$$

Proposition 2.1. *P_m^k is a modular form if $k > 2$.*

Proof. Note that $P_0^k = E_k$, the Eisenstein series of weight k, as we saw in the second talk. If $m > 0$, and for any pair of coprime integers c and d, $\left|(cz+d)^{-k}e^{2\pi i m\gamma z}\right| \le |cz+d|^{-k}$, so $\left|P_m^k(z)\right| \le |E_k(z)|$ and hence P_m^k is a modular form. □

3 The Fourier Expansion of Poincaré Series

In fact, P_m^k is also a cusp form. To see this, we need to calculate its Fourier expansion. Let a_n^k be the n^{th} coefficient of P_m^k. Then by definition

$$\begin{aligned}
a_n^k = a_n &= \int_0^1 P_m^k(z) e^{-2\pi i n z}\,\mathrm{d}z \\
&= \int_0^1 \sum_{\substack{gcd(c,d)=1\\ c\ge 0}} (cz+d)^{-k} e^{2\pi i m\gamma z} e^{-2\pi i n z}\,\mathrm{d}z \\
&= \int_0^1 \sum_{\substack{d=\pm 1\\ c=0}} d^{-k} e^{2\pi i m z} e^{-2\pi i n z}\,\mathrm{d}z + \int_0^1 \sum_{\substack{gcd(c,d)=1\\ c>0}} (cz+d)^{-k} e^{2\pi i m\gamma z} e^{-2\pi i n z}\,\mathrm{d}z
\end{aligned}$$

where in the last step we used that if $c = 0$ and $gcd(c,d) = 1$ the only two valid values for d are ± 1. Now we look at the first term and see that

$$\int_0^1 e^{-2\pi i (m-n) z}\,\mathrm{d}z = \delta_{mn}.$$

So we get

$$a_n^k = 2\delta_{mn} + \int_0^1 \sum_{\substack{gcd(c,d)=1\\ c>0}} (cz+d)^{-k} e^{2\pi i (m\gamma z - n z)}\,\mathrm{d}z.$$

For a fixed $c > 0$ we can write $d = lc + d'$ for $l \in \mathbf{Z}$, $0 \le d' < c$.
It holds that $l \equiv d' \mod c$ and $gcd(c, d) = 1$. So can write

$$a_n = 2\delta_{mn} + \sum_{\substack{c>0 \\ d' \mod c \\ gcd(c,lc+d')=1}} \sum_{l\in\mathbf{Z}} \int_0^1 (c(z+l)+d')^{-k} e^{2\pi i(m\gamma(z+l)-nz)}\,\mathrm{d}z. \quad (1)$$

We want to express $\gamma(z+l)$ in another way. For this we look at the following equation:

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{1}{c(cz+d)}$$

With this we see that

$$\gamma(z+l) = \frac{a}{c} - \frac{1}{c(c(z+l)+d)}. \quad (2)$$

Putting equation (2) in (1) we get:

$$a_n = 2\delta_{mn} + \sum_{\substack{c>0 \\ d \mod c \\ gcd(c,d)=1 \\ ad-bc=1}} \sum_{l\in\mathbf{Z}} \int_0^1 (c(z+l)+d)^{-k} e^{-2\pi i n z} e^{m(\frac{a}{c} - \frac{1}{c(c(z+l)+d)})}\,\mathrm{d}z$$

We use the change of variables $z + l = z'$. Note that $e^{2\pi i n z} = e^{2\pi i n z'}$.

$$\begin{aligned} a_n &= 2\delta_{mn} + \sum_{\substack{c>0 \\ d \mod c \\ gcd(c,d)=1}} \sum_{l\in\mathbf{Z}} \int_l^{l+1} (cz'+d)^{-k} e^{\frac{ma}{c} - \frac{m}{c(cz+d)} - nz}\,\mathrm{d}z' \\ &= 2\delta_{mn} + \sum_{\substack{c>0 \\ d \mod c \\ gcd(c,d)=1}} \int_{-\infty}^{\infty} (cz+d)^{-k} e^{\frac{ma}{c} - \frac{m}{c(cz+d)} - nz}\,\mathrm{d}z \end{aligned}$$

Now we substitute $z' = z + \frac{d}{c}$.

$$a_n = 2\delta_{mn} + \sum_{c>0} c^{-k} \sum_{\substack{c>0 \\ d \mod c \\ gcd(c,d)=1}} e^{\frac{ma}{c}} \int_{-\infty}^{\infty} z^{-k} e^{\frac{-m}{c^2 z} - n(z - \frac{d}{c})} \, \mathrm{d}z$$

$$= 2\delta_{mn} + \sum_{c>0} c^{-k} \sum_{\substack{c>0 \\ d \mod c \\ gcd(c,d)=1}} e^{\frac{ma+nd}{c}} \int_{-\infty}^{\infty} z^{-k} e^{\frac{-m}{c^2 z} - nz} \, \mathrm{d}z$$

Definition 3.1. *The Kloosterman sums are defined as*

$$K(m,n,c) = \sum_{\substack{d \mod c \\ gcd(c,d)=1}} e^{\frac{m\overline{d}+nd}{c}}$$

where $\overline{d}$ *is defined to be the inverse of* d *in* $\left(\mathbf{Z}/_{c\mathbf{Z}} \right)^{\times}$ $(d\overline{d} = 1 \mod c)$

To use this definition in our calculation we need to verify that $a = \overline{d}$. This follows from the equation for the determinante of our matrix $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $ad - bc = 1$. From this we get that $ad \equiv 1 \mod c$ which is the same as saying that $a = \overline{d}$

Definition 3.2. *We define the function* $I(m,n,c,k)$ *to be*

$$I(m,n,c,k) = \int_{-\infty}^{\infty} z^{-k} e^{\frac{-m}{c^2 z} - nz} \, \mathrm{d}z.$$

These two definitions let us write the Fourier coefficient as follows:

$$a_n = 2\delta_{mn} + \sum_{c>0} c^{-k} K(m,n,c) I(m,n,c,k).$$

Definition 3.3. *The Bessel function of order* α *is given by*

$$J_\alpha(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1} (\frac{z}{2})^{2m+\alpha}.$$

We can evaluate the integral $I(m,n,c,k)$ as follows:

Lemma 3.1.

$$I(m,n,c,k) = \begin{cases} 0 & \text{if } n \leq 0 \\ \frac{(2\pi i)^k (-n)^{k-1}}{(k-1)!} & \text{if } m = 0 \\ 2\pi (i)^k (c\sqrt{\frac{n}{m}})^{k-1} J_{k-1}(\frac{4\pi\sqrt{mn}}{c}) & \text{if } m \geq 1 \end{cases}$$

Proof. [2] □

Now we look at the fourier coefficient in the case $m = 0$ which is given by

$$a_n = \frac{(2\pi i)^k}{(k-1)!} n^{k-1} \sum_{\substack{c>0 \\ d \mod c \\ gcd(c,d)=1}} c^{-k} e^{\frac{nd}{c}}.$$

Definition 3.4. *Let $c, n \in \mathbf{Z}$, then we define to Ramanujan sums to be*

$$R_c(n) = \sum_{\substack{d \mod c \\ gcd(c,d)=1}} e^{2\pi i \frac{nd}{c}}.$$

Lemma 3.2. *Let $\sigma_s(n) = \sum_{d|n} d^s$ and let $\zeta(s)$ denote the Riemann-Zeta-Function. Then the following identity holds:*

$$\sum_{c=1}^{\infty} \frac{R_c(n)}{c^s} = \frac{\sigma_{s-1}(n)}{n^{s-1}\zeta(s)}.$$

Proof. [3] □

Using this identity in the above expression of our Fourier coefficient we get

$$a_n = \frac{(2\pi i)^k}{(k-1)!} \frac{\sigma_{k-1}(n)}{\zeta(k)}.$$

Now we want to look at the case $m \geq 1$. In this case $a_0 = 0$ follows directly from the evaluation of the integral $I(m, n, c, k)$. Hence Poincaré series are cusp forms, and $P_m^k = 0$ for $k \in \{4, 6, 8, 10\}$ since we already proved that the discriminant function Δ with degree 12 is the lowest degree cusp form.

Now that we have everything together we can express the n^{th} coefficient of the m^{th} Poincaré series:

$$a_{n,m}^k = \sum_{c>0} \frac{K(m,n,c)}{c} \frac{2\pi}{i^k} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

Theorem 3.3. *Let $f \in \mathcal{S}_k(\Gamma)$ and $P_m^k(z)$ the m^{th} Poincaré series. Then $\langle f, P_m \rangle = c_{k,m} a_m$ where $f(z) = \sum a_m q^m$ and $c_{k,m} = \frac{(k-2)!}{(4\pi m)^{k-1}} = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}$*

We use the so called Rankin-Selberg method to map an integral over ${}_{\Gamma}\backslash^{H}$ to an integral over ${}_{\Gamma_\infty}\backslash^{H}$ which we can understand better because it is just a strip in the complex plane. We unfold the integral which simplifies the fundamental domain.

Proof. We look at the case $m > 0$. Then we get

$$\begin{aligned}
\langle f, P_m \rangle &= \int_{\Gamma\backslash^H} f(z)\overline{P_m(z)} y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\
&= \int_{\Gamma\backslash^H} f(z) \overline{\sum_{\gamma \in \Gamma_\infty\backslash\Gamma} e^{2\pi i m \gamma z}(cz+d)^{-k}} y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\
&= \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \int_{\Gamma\backslash^H} f(z) e^{-2\pi i m(\overline{\gamma z})} \overline{j(\gamma, z)}^{-k} y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\
&= \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \int_{\gamma(\Gamma\backslash^H)} f(\gamma^{-1} z) e^{-2\pi i m \overline{z}} \overline{j(\gamma, \gamma^{-1} z)}^{-k} (Im(\gamma^{-1} z))^k \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\
&= \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \int_{\gamma(\Gamma\backslash^H)} j(\gamma^{-1}, z)^k f(z) e^{-2\pi i m \overline{z}} \overline{j(\gamma^{-1}, z)}^{k} \frac{y^k}{|j(\gamma^{-1}, z)|^{2k}} \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\
&= \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \int_{\gamma(\Gamma\backslash^H)} f(z) e^{-2\pi i m \overline{z}} y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2} \\
&= \int_{\Gamma_\infty\backslash^H} f(z) e^{-2\pi i m x} e^{-2\pi m y} y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2}
\end{aligned}$$

The action Γ_∞ on $\mathbb{H}$ is generated by the translation $z \mapsto z + 1$ and hence the fundamental domain is defined by $0 \leq x \leq 1$. So we find

$$\langle f, P_m \rangle = \int_0^\infty \int_0^1 f(z) e^{-2\pi i m x} e^{-2\pi m y} y^k \frac{\mathrm{d}x\mathrm{d}y}{y^2}$$

Now we write $f(z) = \sum_{n=1}^\infty a_n e^{2\pi i n z}$ and notice that we can exchange the sum and the integral by Fubini.

$$\begin{aligned}
\langle f, P_m \rangle &= \sum_{n=1}^\infty a_n \int_0^\infty e^{-2\pi(m+n)y} y^k \frac{\mathrm{d}y}{y^2} \underbrace{\int_0^1 e^{2\pi i(n-m)x} \mathrm{d}x}_{\delta_{mn}} \\
&= a_m \int_0^\infty e^{-4\pi m y} y^k \frac{\mathrm{d}y}{y^2} \\
&= a_m (4\pi m)^{1-k} \Gamma(k-1)
\end{aligned}$$

In the case $m = 0$ we first need to extend the definition of the Peterssen inner product to entire modular forms in one component. This can be done by using the same definition as before and we can check that this is well-defined. Then we can see $\langle f, P_0^k \rangle = a_0 = 0$. Also note that $P_0^k = E_k$. So we get that $\langle f, E_k \rangle = 0$ and it follows that f and E_k are orthogonal.

□

Corollary 3.3.1. *Let $k \geq 3$ then $\{P_m^k(z) | m \geq 1\}$ generate $\mathcal{S}_k(\Gamma)$.*

Proof. Let M be the subspace of $\mathcal{S}_k(\Gamma)$ generated by P_m^k and $M^\perp$ the orthogonal complement. Take $f \in M^\perp$ and by theorem 3.2 we get $\langle f, P_m \rangle = c_{k,m} a_m = 0$, $\forall m$. From this we get that $f \equiv 0$.
Since f was an arbitrary element in $M^\perp$ we get that $\mathcal{S}_k(\Gamma) = M$.

□

Example 3.1. *The discriminant function $\Delta(z)$ is a cusp form of weight 12 as seen earlier. We can express it as follows:*

$$\Delta(z) = e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^{24} = \sum_{n=1}^{\infty} \underbrace{\tau(n)}_{n^{th} \text{fourier coeff.}} e^{2\pi i n z}$$

Using the previous theorem for Δ we get $\langle \Delta, P_m^{12} \rangle = \frac{10! \tau(m)}{(4\pi m)^{11}}$.

Since $\mathcal{S}_{12}(\Gamma)$ is 1-dimensional, P_m^{12} is a scalar multiple of Δ. Hence we can find $P_m^{12}(z) = \frac{10! \tau(m)}{(2\pi m)^{11} \langle \Delta, \Delta \rangle} \Delta(z)$ by using the fourier expansion of Δ.

Additive:

Theorem 3.4. *(Petersson trace formula)*
Let $\mathcal{F}$ be an orthonormal basis of $\mathcal{S}_k(\Gamma_0(N))$. Then

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}} \sum_{f \in \mathcal{F}} \overline{a_m} a_n = \delta_{mn} + \frac{1}{(2\pi i)^k} \sum_{\substack{c>0 \\ N|c}} \frac{K(m,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

Proof. [4] □

Using the Petersson trace formula we see that $P_m^{12} = 0 \iff a_m = 0$
Now there exists the very famous Lehmer conjecture which states that $a_m \neq 0 \, \forall m \geq 1$. This has been verified for $m < 22689242781695999$ by Jordan and Kelly [5] but has not been proven yet.

References

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