HECKE OPERATORS AND L-FUNCTIONS OF MODULAR FORMS

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1. Motivation

We want to define Hecke operators, which are linear maps, indexed by $m \in \mathbb{N}$ and sending modular forms of a given weight k to modular forms of the same weight and cusp forms to cusp forms:

$$T_m : M_k(SL(2,\mathbb{Z})) \to M_k(SL(2,\mathbb{Z})),$$
$$T_m : S_k(SL(2,\mathbb{Z})) \to S_k(SL(2,\mathbb{Z})).$$

Hecke operators enable us to apply methods from linear algebra into the theory of modular forms: Hecke operators are pairwise commuting and self-adjoint on $S_k(SL(2,\mathbb{Z}))$, where the hermitian form is taken to be the Petersson inner product. Hence by Spectral Theorem we have the existence of an orthonormal basis of $S_k(SL(2,\mathbb{Z}))$ consisting of simultaneous eigenfunctions (called Hecke eigenforms) of all of the operators T_m, $m \in \mathbb{N}$.

2. Definition of Hecke Operators and their first properties

Fix $m \in \mathbb{N}$ and consider the space: $M_m = \{A \in Mat(2,\mathbb{Z}) \mid det\, A = m\}$. We have an action of $SL(2,\mathbb{Z})$ on M_m by left-multiplication:

$$SL(2,\mathbb{Z}) \times M_m \to M_m.$$

Hence we have a decomposition of M_m into orbits. Hecke operators can be thought of as averaging operators on the space of modular forms, where the sum is taken over orbit representatives of the above action. Before defining the Hecke operators we want to study the above action and give explicitly a set of orbit representatives of the group action. This will be done in the following proposition:

Proposition 2.1. *A set of orbit representatives of the above action is given by the following matrices:*

$${M_m}' = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad = m,\ 0 \le b \le d-1,\ d > 0\}.$$

Moreover two different elements of ${M_m}'$ are inequivalent, meaning that each element gives rise to a distinct orbit.

Proof. First we want to show that each element $A \in M_m$ appears in some orbit. Hence we want to find $\gamma \in SL(2,\mathbb{Z})$ and $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in {M_m}'$ such that $\gamma^{-1}A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. First let $A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in M_m$ and set: $a = (x_1, x_3)$, $\gamma_1 = \frac{x_1}{a}$, $\gamma_3 = \frac{x_3}{a}$. Hence $(\gamma_1, \gamma_3) = 1$ and by lemma of Bézout we can find $\gamma_2, \gamma_4 \in \mathbb{Z}$ such that $\gamma_1\gamma_4 - \gamma_2\gamma_3 = 1$. Hence $\begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix} \in SL(2,\mathbb{Z})$. Now consider the following calculations:

$$\begin{pmatrix}\gamma_1 & \gamma_2\\ \gamma_3 & \gamma_4\end{pmatrix}^{-1}\begin{pmatrix}x_1 & x_2\\ x_3 & x_4\end{pmatrix} = \begin{pmatrix}\gamma_4x_1-\gamma_2x_3 & \gamma_4x_2-\gamma_2x_4\\ \gamma_1x_3-\gamma_3x_1 & \gamma_1x_4-\gamma_3x_2\end{pmatrix} = \begin{pmatrix}a & \gamma_4x_2-\gamma_2x_4\\ 0 & \gamma_1x_4-\gamma_3x_2\end{pmatrix}.$$

This follows from:
$\gamma_4x_1-\gamma_2x_3=\gamma_4\gamma_1a-\gamma_2\gamma_3a=a(\gamma_1\gamma_4-\gamma_2\gamma_3)=a.$
$\gamma_1x_3-\gamma_3x_1=\gamma_1\gamma_3a-\gamma_3\gamma_1a=0.$
Now set: $b'=\gamma_4x_2-\gamma_2x_4,\ \ d=\gamma_1x_4-\gamma_3x_2.$
Hence we have $\gamma^{-1}A=\begin{pmatrix}a & b'\\ 0 & d\end{pmatrix}$. We can ensure that $d>0$ by multiplying from the left by the matrix $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$. By the division algorithm we can also find $b=b'-kd$, where $k\in\mathbb{Z}$ and $0\le b\le d-1$ then we just multiply from the left by $\begin{pmatrix}1 & -k\\ 0 & 1\end{pmatrix}$ to obtain the matrix with the desired properties: $\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}$. It remains to prove that $ad=m$. This follows by simple calculations or by observing that $\gamma'A=\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}$, for some $\gamma'\in SL(2,\mathbb{Z})$ and that the determinant of A equals m.

Now we want to prove the second claim, namely that different elemens in ${M_m}'$ give different orbits. Let $\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}, \begin{pmatrix}a' & b'\\ 0 & d'\end{pmatrix}\in {M_m}'$. Assume there is $\gamma\in SL(2,\mathbb{Z})$ such that

$$\gamma\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}=\begin{pmatrix}a' & b'\\ 0 & d'\end{pmatrix}.$$

This means that

$$\begin{pmatrix}a\gamma_1 & b\gamma_1+d\gamma_2\\ a\gamma_3 & a\gamma_3+d\gamma_4\end{pmatrix}=\begin{pmatrix}a' & b'\\ 0 & d'\end{pmatrix}.$$

This implies that $\gamma_3=0$, but then since $\gamma_1\gamma_4=1$, we have that: $\gamma_1,\gamma_4=1 \text{ or} -1$. Since $d,d'>0$, we have that $\gamma_1=\gamma_4=1$. But then $d=d'$ and $0\le b,b'\le d-1$ implies that $-d+1\le b-b'\le d-1$. But $b-b'=d\gamma_2$. Hence $\gamma_2=0$. Hence γ is necessarily the identity matrix and $\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}=\begin{pmatrix}a' & b'\\ 0 & d'\end{pmatrix}$. The converse implication proves the claim. □

Before defining the Hecke operators we need to recollect some definitions:

Definition 2.2. Let $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in GL_2^+(\mathbb{R})$ and $z\in\mathbb{H}$. Define $j(\gamma,z)=cz+d$. Now let $f:\mathbb{H}\to\mathbb{C}$ and an integer k fixed. Define the function[1]

$$f[\gamma]_k:\mathbb{H}\to\mathbb{C},$$

$$f[\gamma]_k(z)=(det\,\gamma)^{k/2}j(\gamma,z)^{-k}f(\gamma z).$$

This function has the following properties:

$$f[\gamma_1\gamma_2]_k = f[\gamma_1]_k[\gamma_2]_k\ \ \forall\gamma_1,\gamma_2\in GL_2^+(\mathbb{R}),$$
$$f\mapsto f[\gamma]_k \text{ is a linear operator } \forall\gamma\in GL_2^+(\mathbb{R}).$$

[1]In literature one often finds the following notation for this function: $f|_k\gamma$.

Remark 2.3. In general we cannot expect $f[A]_k$ to be a modular form for arbitrary $A \in GL_2^+(\mathbb{R})$ and $f \in M_k(SL(2,\mathbb{Z}))$. Hecke operators (which will be defined below) are special in that regard: By summing elements of the form $f[A]_k$ for a given modular form f we indeed get back a modular form!

Now we are ready to define the Hecke operators:

Definition 2.4. Let $m \in \mathbb{N}$. We define the m-th Hecke operator T_m as follows:
For $f \in M_k(SL(2,\mathbb{Z}))$ let

$$T_m f := m^{k/2-1} \sum_{A \in M_m/\sim} f[A]_k. \tag{2.1}$$

The sum is meant to be taken over arbitrary orbit representatives.[2] After showing that the above statement is well-defined, i.e independent of the choice of orbit representatives, we can insert our distinguished set of orbit representatives from proposition 2.1 to obtain:

$$T_m f = m^{k/2-1} \sum_{\substack{ad=m \\ d>0}} \sum_{b=0}^{d-1} f\left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right]_k.$$

For $z \in \mathbb{H}$ we have:

$$T_m f(z) = m^{k-1} \sum_{\substack{ad=m \\ d>0}} \sum_{b=0}^{d-1} d^{-k} f\Big(\frac{az+b}{d}\Big).$$

Lemma 2.5. *The Hecke operators are well-defined, i.e the definition does not depend on the choice of orbit representatives.*

Proof. We want to show now that the Hecke operators are well-defined. For this let N be the (finite) number of orbits and Let $\{A_i \mid i = 1,\dots,N\}, \{A'_i \mid i = 1,\dots,N\}$ be two arbitrary orbit representatives. Then up to reordering of the indices we have that $A_i = \gamma_i A'_i$ for some $\gamma_i \in SL(2,\mathbb{Z}), i = 1,\dots,N$. Then the claim follows directly from the weak modularity of f:

$$\sum_{i=1}^{N} f[A_i]_k = \sum_{i=1}^{N} f[\gamma_i A'_i]_k = \sum_{i=1}^{N} f[\gamma_i]_k[A'_i]_k = \sum_{i=1}^{N} f[A'_i]_k.$$

□

Theorem 2.6. *Let $m \in \mathbb{N}$, $f \in M_k(SL(2,\mathbb{Z}))$. Then $T_m f : \mathbb{H} \to \mathbb{C}$ is*
(i) holomorphic on $\mathbb{H}$,
(ii) holomorphic at ∞,
(iii) weakly modular of weight k on $SL(2,\mathbb{Z})$.
Hence $T_m f \in M_k(SL(2,\mathbb{Z}))$ and $T_m : M_k(SL(2,\mathbb{Z})) \to M_k(SL(2,\mathbb{Z}))$ is a linear operator *on the* space of modular forms of weight k.

Proof. The first claim is obviously true. The second claim will follow from the next lemma, where we will discuss the action of T_m on the Fourier expansion of $f \in M_k(SL(2,\mathbb{Z}))$. Let us prove the third claim. Given $f \in M_k(SL(2,\mathbb{Z}))$ we must show that:

$$T_m f[\gamma]_k = T_m f \quad \forall \gamma \in SL(2,\mathbb{Z}).$$

[2]This notation may seem ambiguous, since the elements in $M_m/\sim$ are orbits. However this should not lead to any confusion.

Consider the following:

$$T_m f[\gamma]_k = \Big(m^{k/2-1} \sum_{A\in M_m/\sim} f[A]_k\Big)[\gamma]_k = m^{k/2-1} \sum_{A\in M_m/\sim} f[A\gamma]_k = T_m f.$$

The second equality follows from known properties. The last equality follows from the fact that $\{A\gamma \mid A \in M_m/\sim\}$ is another set of orbit representatives. $\square$

To prove holomorphicity at ∞, we need to know how the Hecke operator acts on the Fourier expansions of modular forms. For that let us consider the next lemma:

Lemma 2.7. *Let $f \in M_k(SL(2,\mathbb{Z}))$ have the* Fourier expansion*:*

$$f(z) = \sum_{n=0}^{\infty} f_n e^{2\pi i n z}.$$

Then the Fourier expansion *of $T_m f$ is given by the following formula:*

$$T_m f(z) = \sum_{n=0}^{\infty} \Bigg(\sum_{a|n,m} a^{k-1} f_{\frac{mn}{a^2}} \Bigg) e^{2\pi i n z}.$$

Hence $T_m f$ is holomorphic at ∞.

Proof. We need the following easy result for the proof:

$$\sum_{b=0}^{d-1} e^{2\pi i n b/d} = \begin{cases} d, & \text{if } d \mid n \\ 0, & \text{otherwise} \end{cases}.$$

Let $f(z) = \sum_{n=0}^{\infty} f_n e^{2\pi i n z}$ be the Fourier expansion. Applying the Hecke operator we obtain:

$$\begin{aligned} T_m f(z) &= m^{k-1} \sum_{\substack{ad=m \\ d>0}} \sum_{b=0}^{d-1} d^{-k} \sum_{n=0}^{\infty} f_n e^{2\pi i n \frac{az+b}{d}} = \frac{1}{m} \sum_{\substack{ad=m \\ d>0}} \sum_{b=0}^{d-1} \Big(\frac{m}{d}\Big)^k \sum_{n=0}^{\infty} f_n e^{2\pi i n \frac{az}{d}} e^{2\pi i n \frac{b}{d}} \\ &= \sum_{\substack{ad=m \\ d>0}} \Big(\frac{m}{d}\Big)^{k-1} \sum_{\substack{n=0 \\ d|n}}^{\infty} f_n e^{2\pi i n \frac{az}{d}} = \sum_{\substack{ad=m \\ d>0}} \Big(\frac{m}{d}\Big)^{k-1} \sum_{n=0}^{\infty} f_{nd} e^{2\pi i n a z} \\ &= \sum_{a|m} a^{k-1} \sum_{n=0}^{\infty} f_{\frac{nm}{a}} e^{2\pi i n a z} = \sum_{r=0}^{\infty} \sum_{\substack{a|m \\ a|r}} a^{k-1} f_{\frac{rm}{a^2}} e^{2\pi i r z}. \end{aligned}$$

$\square$

Corollary 2.8. *If $f \in S_k(SL(2,\mathbb{Z}))$, then $T_m f \in S_k(SL(2,\mathbb{Z}))$. Hence the Hecke operators maps cusp forms to cusp forms and we have the restriction map:*

$$T_m : S_k(SL(2,\mathbb{Z})) \to S_k(SL(2,\mathbb{Z})).$$

The next goal is to show that the Hecke operators T_m form a commuting family of endomorphisms on $M_k(SL(2,\mathbb{Z}))$ and $S_k(SL(2,\mathbb{Z}))$.

Lemma 2.9. *If $(m,n) = 1$, then $T_m T_n = T_{mn} = T_n T_m$.*

Proof. Let $f \in M_k(SL(2,\mathbb{Z}))$. Let $f(z) = \sum_{r=0}^{\infty} f_r e^{2\pi i r z}$ be the Fourier expansion. Consider the mn-th Hecke operator T_{mn}. It has the following Fourier expansion:

$$T_{mn}f(z) = \sum_{r=0}^{\infty}\left(\sum_{t|r,mn} t^{k-1} f_{\frac{rmn}{t^2}}\right)e^{2\pi i r z}.$$

Now there is a bijection (exercise) between the tuples of divisors $\{(a,b) \mid a|r, b|r, a|n\ b|m\}$ and the divisors $\{t \mid t|r, t|mn\}$ given by $(a,b) \mapsto ab$. (Here we use the first time that $(n,m) = 1$). Hence we can rewrite the above sum in the following way:

$$T_{mn}f(z) = \sum_{r=0}^{\infty}\left(\sum_{ab|r,mn} a^{k-1}b^{k-1} f_{\frac{rmn}{a^2b^2}}\right)e^{2\pi i r z} = \sum_{r=0}^{\infty}\left(\sum_{b|r,m}\sum_{a|r,n} a^{k-1}b^{k-1} f_{\frac{rmn}{a^2b^2}}\right)e^{2\pi i r z}.$$

Now consider the n-th Hecke operator T_n. It has the following Fourier expansion:

$$T_n f(z) = \sum_{r=0}^{\infty}\left(\sum_{a|r,n} a^{k-1} f_{\frac{rn}{a^2}}\right)e^{2\pi i r z}.$$

Now set $\tilde{f} = T_n f$ and apply the m-th Hecke operator. This gives:

$$T_m(T_n f)(z) = T_m\tilde{f}(z) = \sum_{r=0}^{\infty}\left(\sum_{b|r,m} b^{k-1} \tilde{f}_{\frac{rm}{b^2}}\right)e^{2\pi i r z}.$$

Now notice that $\tilde{f}_k$ is the k-th Fourier-coefficient of $T_n f$. Hence:

$$\tilde{f}_{\frac{rm}{b^2}} = \sum_{a|\frac{rm}{b^2},n} a^{k-1} f_{\frac{rmn}{b^2a^2}}.$$

Now inserting the Fourier-coefficient into the above sum we get:

$$\sum_{r=0}^{\infty}\left(\sum_{b|r,m}\sum_{a|\frac{rm}{b^2},n} a^{k-1}b^{k-1} f_{\frac{rmn}{a^2b^2}}\right)e^{2\pi i r z}.$$

But now consider the following:
$a|\frac{rm}{b^2}$ implies that $a|rm$. By assumption $a|n$. Hence by using for the second time that $(n,m) = 1$, we have that $(a,m) = 1$. Then $a|rm$ implies that $a|r$. (This can be proven by lemma of Bézout).
Hence the above sum simplifies to:

$$\sum_{r=0}^{\infty}\left(\sum_{b|r,m}\sum_{a|r,n} a^{k-1}b^{k-1} f_{\frac{rmn}{a^2b^2}}\right)e^{2\pi i r z}.$$

What we have shown is that for arbitrary $f \in M_k(SL(2,\mathbb{Z}))$ it holds that $T_m T_n f = T_{mn} f$. Now the claim follows easily from the commutativity of the natural numbers:

$$T_m T_n = T_{mn} = T_{nm} = T_n T_m.$$

□

Lemma 2.10. *Let p be a prime number. Then we have the following:*

$$\forall r, s \in \mathbb{N} : \ T_{p^r} T_{p^s} = T_{p^s} T_{p^r}.$$

Proof. See for example: [2] or see [1] for a different approach. ☐

Theorem 2.11. *The Hecke operators commute:*

$$\forall m, n \in \mathbb{N}: \ T_m T_n = T_n T_m.$$

Proof. Let $m, n \in \mathbb{N}$. By prime factorization, put: $n = p_1^{\alpha_1} \dots p_j^{\alpha_j}$, $m = q_1^{\beta_1} \dots q_k^{\beta_k}$. Then:

$$T_n T_m = T_{p_1^{\alpha_1}} \dots T_{p_j^{\alpha_j}} T_{q_1^{\beta_1}} \dots t_{q_k^{\beta_k}}.$$

Now, if $p_i = q_l$ for some i, l, we can commute them by lemma 2.10. If they are not equal, we can commute them by lemma 2.9. Hence the claim follows. ☐

We will state an important property of the Hecke operators on $S_k(SL(2,\mathbb{Z}))$, namely that they are self-adjoint w.r.t the Petersson inner product:

Theorem 2.12. *The Hecke operators $T_m : S_k(SL(2,\mathbb{Z})) \to S_k(SL(2,\mathbb{Z}))$ are self-adjoint w.r.t the Petersson inner product, i.e*

$$\langle T_m f, g \rangle = \langle f, T_m g \rangle \quad \forall\, f, g \in S_k(SL(2,\mathbb{Z})).$$

Proof. First one shows that the statement is true for Poincaré-series. Then one notices that the Poincaré-series span $S_k(SL(2,\mathbb{Z}))$ (last talk). See for example: [6], theorem 6.12. ☐

3. Dirichlet series associated to modular forms

Each modular form $f \in \mathcal{M}_k(SL(2,\mathbb{Z}))$ has an associated Dirichlet series, its *L-function*. Let $f(z) = \sum_{n=0}^{\infty} a^n e^{2\pi i n z}$ be its Fourier expansion, let $s \in \mathbb{C}$ be a complex variable, and write formally

$$L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

Convergence of $L(s, f)$ in a half plane of s-values follows from estimating the Fourier coefficents of f. Note that the Dirichlet series begains from 1.

Proposition 3.1. *If $f \in \mathcal{M}_k(SL(2,\mathbb{Z})))$ is a cusp form then its fourier coefficients satisfy $a_n = O(n^{k/2})$. If f is not a cusp form then its fourier coeffients satisfy $a_n = O(n^k)$.*

Proof. let $q = e^{2\pi i \tau} = e^{2\pi i (x+iy)}$, let $g(q) = \sum_{n=1}^{\infty} a_n q^n$, a holomorphic function on the unit disk $q : |q| < 1$, then for any $r \in (0, 1)$, we have

$$a_n = \frac{1}{2\pi i} \int_{|q|=r} g(q) q^{-n} dq/q = \int_{x=0}^{1} f(x+iy) e^{-2\pi i n (x+iy)} dx$$

for any $y > 0$, and letting $y = 1/n$

$$a_n = e^{2\pi} \int_{x=0}^{1} f(x + i/n) e^{-2\pi i n x} dx$$

Since f is a cusp form, we know that it decays rapidly when it approaches the cusp. Thus $\mathrm{Im}(\tau)^{k/2} |f(\tau)|$ is bounded on the upper half plane, so estimating the last integral, we kown that $|a_n| \le C n^{k/2}$, then we get the result.

If E_k is an Eisenstein series in $\mathcal{M}_k(SL(2,\mathbb{Z}))$, its nth Fourier coefficient of f is $\sigma_{k-1}(n)$, and

$$\sigma_{k-1} = \sum_{d|n} d^{k-1} = \sum_{d|n} (\frac{n}{d})^{k-1} = n^{k-1} \sum_{d|n} \frac{1}{d^{k-1}} < n^{k-1}\zeta(k-1)$$

we have that $|a_n| \leq Cn^{k-1}$. Since every modular form is the sum of a cusp form and an Eisenstein series, the rest follows. □

Corollary 3.2. *if $f \in \mathcal{M}_k(SL(2,\mathbb{Z}))$ is a cusp form then $L(s,f)$ converges absultely for all s with $\Re(s) > k/2+1$. If f is not a cusp form then $L(s,f)$ converges absolutely for all s with $\Re(s) > k$*

Remark 3.3. Daniel Bump in [3] remarked that: The estimate $|a_n| \leq Cn^{k/2}$ called the *trival estimate*, is due to Hardy (1927) and (more simply) Hecke (1937). The correct estimate $|a_n| \leq Cn^{(k-1)/2+\epsilon}$ for any $\epsilon > 0$ was conjectured (for $f = \Delta$) by Ramanujan (1916); this famous statement, the *Ramanujan conjecture*, was finally proved around 1970 by Deligne(1971) using difficult techniques from algebraic geometry.

In order to estimate the convergence of some integrals which we will use later, we need the following lemma

Lemma 3.4. *For a sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers, for $z \in \mathbb{H}$, put*

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}, \tag{3.1}$$

and $a_n = O(n^v)$ with some $v > 0$. Then the right-hand side of equation 3.1 convergent absolutely and uniformly on any compact subset of $\mathbb{H}$, and $f(z)$ is holomorphic on $\mathbb{H}$. Moreover,

$$f(z) = O(\mathrm{Im}(z)^{-v-1}) \qquad (\mathrm{Im}(z) \to 0)$$
$$f(z) - a_0 = O(e^{-2\pi\,\mathrm{Im}(z)}) \qquad (\mathrm{Im}(z) \to \infty)$$

uniformly on $\Re(z)$

Proof. By the formula

$$\Gamma(v+1) = \lim_{n\to\infty} \frac{n!n^{v+1}}{(v+1)(v+2)\dots(v+n+1)},$$

we have for $v > 0$,

$$\lim_{n\to\infty} n^v/(-1)^n \binom{-v-1}{n} = \Gamma(v+1),$$

And because $a_n = O(n^v)$, there exisits $L > 0$ such that

$$|a_n| \leq L(-1)^n \binom{-v-1}{n}$$

for all $n \geq 0$. Put $z = x + iy$, then we have

$$\sum_{n=0}^{\infty} |a_n||e^{2\pi i n z}| \leq L(\sum_{n=0}^{\infty} (-1)^n \binom{-v-1}{n} e^{-2\pi n y})$$
$$= L(1 - e^{-2\pi y})^{-v-1}$$

Because $(1 - e^{-2\pi y}) = O(y)$ as $y \to 0$, we see that $|f(z)| = O(y^{-v-1})$, also we have $f(z)$ is bounded when $y \to \infty$.

As $f(z) - a_0 = e^{2\pi inz}(\sum_{n=0}^{\infty} a_{n+1}e^{2\pi inz})$, and $\sum_{n=0}^{\infty} a_{n+1}e^{2\pi inz}$ also satisfy the assumption of the lemma, by the conclusion we just proved, it is bounded when $y \to \infty$. Therefore, we have as $y \to \infty$

$$f(z) - a_0 = e^{2\pi inz}(\sum_{n=0}^{\infty} a_{n+1}e^{2\pi inz}) = O(e^{-2\pi y})$$

□

Proposition 3.5. *The* L-function $L(s, f)$ *has meromorphic continuation to all* s *and satisfies a functional equation. In fact, if*

$$\Lambda(s, f) = (2\pi)^{-s}\Gamma(s)L(s, f)$$

then $\Lambda(s, f)$ *extends to an analytic function of* s *if* f *is a cusp form; if it is not a cusp form, then it has simple poles at* $s = 0$ *and* $s = k$*. In general, it satisfies*

$$\Lambda(s, f) = (-1)^{k/2}\Lambda(k - s, f)$$

Proof. By the lemma 3.4, we know that for $t > 0$ and $\Re(s) > k + 1$, $\sum_{n=1}^{\infty} a_n e^{-2\pi nt}$ and $\sum_{n=1}^{\infty}\int_0^{\infty} a_n t^s e^{-2\pi nt}t^{-1}dt$ converges absolutely.

For $\Re(s) > k + 1$, we have

$$\begin{aligned}
\Lambda(s, f) &= (2\pi)^{-s}\Gamma(s)L(s, f) \\
&= \sum_{n=1}^{\infty} a_n(2\pi n)^{-s}\int_0^{\infty} e^{-t}t^{s-1}dt \\
&= \sum_{n=1}^{\infty}\int_0^{\infty} a_n t^s e^{-2\pi nt}t^{-1}dt \\
&= \int_0^{\infty} t^s(\sum_{n=1}^{\infty} a_n e^{-2\pi nt})t^{-1}dt \\
&= \int_0^{\infty} t^s(f(it) - a_0)t^{-1}dt \\
&= -\frac{a_0}{s} + \int_1^{\infty} t^{-s}f(i/t)t^{-1}dt + \int_1^{\infty} t^s(f(it) - a_0)t^{-1}dt
\end{aligned} \tag{3.2}$$

Because f is a modular form, we have $f(it) = (-1)^{k/2}t^{-k}f(i/t)$, then

$$\Lambda(s; f) = -\frac{a_0}{s} - \frac{(-1)^{k/2}a_0}{k - s} + (-1)^{k/2}\int_1^{\infty} t^{k-s}(f(it) - a_0)t^{-1}dt + \int_1^{\infty} t^s(f(it) - a_0)t^{-1}dt$$

By the lemma 3.4, we have

$$f(it) - a_0 = O(e^{-2\pi t}),$$

so that

$$\int_1^{\infty} t^{k-s}(f(it) - a_0)t^{-1}dt$$

and

$$\int_1^{\infty} t^s(f(it) - a_0)t^{-1}dt$$

converges absolutely and uniformly on any vertical strip. Therefore they are holomorphic on the whole s-plane. If we define $\Lambda(s, f)$ for any $s \in \mathbb{C}$ by the integral, it is then a meromorphic function on the whole s-plane. The functional equation follows. □

Now we consider the Euler property of the *L-function.*

As Daniel Bump remarks in the [3]: the first historical hint that a Euler product should be associated with the L-series of a modular form came from Ramanujan's investigation of Δ. The Fourier coefficients of Δ comprise Ramanujan's tau function: $\Delta(z) = \sum \tau(n)q^n$. Ramanujan (1916) conjectured, and Mordell (1917) proved shortly afterward, that

$$\sum_{n=1}^{\infty} \tau(n)n^{-s} = \prod_p (1 - \tau(p)p^{-s} + p^{11-2s})^{-1}.$$

The true explanation of this identity requires the theory of *Hecke operators.*

The Hecke algebra is a commutative family of self-adjoint operators on the finite-dimensional vector space $S_k(SL_2(Z))$, by the spectral theorem, we konw there exists a basis of functions which are eigenfunction of all the Hecke operators. We call them the *Hecke eigenform.*

We will show that these eigenforms has the property of Euler product. And also it is interesting to notice that after some normalization. We can find that the eigenvalues of the Hecke operator are hidden in the Fourier coeffients of the so called normalized Hecke eigenform.

Proposition 3.6. *Now we suppose that f is a Hecke eigenform. Let f_n denotes its Fourier coefficients. Then f_n satisfy the following:*

(1) $f_1 \neq 0$
(2) if $f_1 = 1$, *then* $\lambda(n) = f_n$ *for all* n
(3) if $f_1 = 1$ *then the coeffients* f_n *are multiplicative; that is, if* $(m, n) = 1$, *we have* $f_{mn} = f_m f_n$

Proof. By lemma 2.7 , we have that

$$\lambda(n)f_m = \sum_{a|n,a|m} a^{k-1} f_{\frac{mn}{a^2}} \tag{3.3}$$

Suppose that $(m, n) = 1$. Then the only a that divides both m and n is $a = 1$, so it reduces to

$$\lambda(n)f_m = f_{nm} \tag{3.4}$$

Taking $m = 1$, this implies that $f_n = \lambda(n)f_1$, then we get the conclusion. □

Thus we can adjust the coeffient of the Hecke eigenform by making $f_1 = 1$, such a Hecke eigenform will be called *normalized.*

Theorem 3.7. *If f is a normalized Hecke eigenform, then*

$$L(s, f) = \sum_{n\geq 1} f_n n^{-s} = \prod_p (1 - f_p p^{-s} + p^{k-1-2s})^{-1} \tag{3.5}$$

Proof. Because the coeffients of f is multiplicative, then we have

$$L(s, f) = \sum_{n\geq 1} f_n n^{-s} = \prod_p (\sum_{r=0}^{\infty} f_{p^r} p^{-rs})$$

In order to prove equation 3.5, we only need the equity

$$(1 - f_p p^{-s} + p^{k-1-2s})(\sum_{r=0}^{\infty} f_{p^r} p^{-rs}) = 1$$

To prove this, first observe that by lemma 2.7, we have

$$\lambda(n)f_m = \sum_{a|n,a|m} a^{k-1} f_{\frac{mn}{a^2}}$$

by taking $m = p^r, n = p$, then we have

$$f_{p^{r+1}} - f_p f_{p^r} + p^{k-1} f_{p^{r-1}} = 0$$

Multipling by X^{r+1}, and summing this equation for $r \geq 1$, we have

$$\sum_{r\geq 1} f_{p^{r+1}} X^{r+1} - \sum_{r\geq 1} (f_p X)(f_{p^r} X^r) + \sum_{r\geq 1} (p^{k-1} X^2)(f_{p^{r-1}} X^{r-1}) = 0$$

by adding the corresponding term, we have

$$\sum_{r\geq 0} f_{p^r} X^r - \sum_{r\geq 0} (f_p X)(f_{p^r} X^r) + \sum_{r\geq 0} (p^{k-1} X^2)(f_{p^r} X^r) = 1$$

that is

$$(1 - f_p X + p^{k-1} X^2)(\sum_{r=0}^{\infty} f_{p^r} X^r) = 1$$

taking $X = p^{-s}$, we get the Euler product.

□

This section is based on [3] and [5].

4. Hecke's converse theorem

We first write the following condition for conveience to quote later.

Condition 4.1. let $f(z)$ be a function on $\mathbb{H}$, $f(z)$ has a fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z},$$

which converges absolutely and uniformly on any compact subset of $\mathbb{H}$. And there exists $v > 0$ such that:

$$f(z) = O(\mathrm{Im}(z)^{-v}), (\mathrm{Im}(z) \to 0),$$

uniformly on $\Re(z)$.

If f is holomorphic on $\mathbb{H}$, satisfies the condition 4.1, then by a similar proof as propostion 3.1, we have that $a_n = O(n^v)$.

Remark 4.2. By the above lemma 3.4, all holomorphic functions $f(z)$ on $\mathbb{H}$ satisfying the condition 4.1 correspond bijectively to all sequences $\{a_n\}_{n=1}^{\infty}$ of complex numbers such that $a_n = O(n^v)$ with $v > 0$.

Theorem 4.3 (Phragmen-Lindelöf)**.** *For two real numbers v_1, v_2 $v_1 < v_2$, put*

$$F = \{s \in \mathbb{C} | v_1 \leq \Re(s) \leq v_2\}.$$

Let ϕ be a holomorphic function on a domain contating F satisfying

$$|\phi(s)| = O(e^{|\tau|^{\delta}}) \ (|\tau| \to \infty), \quad s = \sigma + i\tau$$

uniformly on F with $\delta > 0$. For real number b, if on $\Re(s) = v_1$ and $\Re(s) = v_2$,

$$|\phi(s)| = O(|\tau|^{\delta}) \ (|\tau| \to \infty)$$

then uniformly on F, we have:

$$|\phi(s)| = O(|\tau|^\delta) \ (|\tau| \to \infty)$$

Proof. By assumption, there exists $L > 0$ such that $|\phi(s)| \le Le^{|\tau|^\delta}$. First consider the case when $b = 0$. Then there exist $M > 0$, such that $|\phi(s)| \le M$ on the lines $\Re(s) = v_1$ and $\Re(s) = v_2$. Let m be a positive integer such that $m \equiv 2 \mod 4$. Put $s = \sigma + i\tau$. Since $\Re(s^m) = \Re((\sigma + i\tau)^m)$ is a polynomial of σ and τ, and the highest term of τ is $-\tau^m$, we have

$$\Re(s^m) = -\tau^m + O(|\tau|^{m-1}) \ (|\tau| \to \infty),$$

uniformly on F, so that $\Re(s^m)$ has an upper bound on F. Taking m and N so that $m > \delta$ and $\Re(s^m) \le N$, we have, for any $\epsilon > 0$, on $\Re(s) = v_1$ and $\Re(s) = v_2$,

$$|\phi(s)e^{\epsilon s^m}| \le Me^{\epsilon N},$$

and

$$|\phi(s)e^{\epsilon s^m}| = O(e^{|\tau|^\delta - \epsilon\tau^m + K|\tau|^{m-1}}) \to 0, \ (|\tau| \to \infty)$$

uniformyly on F. By the maximum principle, we see

$$|\phi(s)e^{\epsilon s^m}| \le Me^{\epsilon N}, \ (s \in F)$$

Letting $\epsilon \to 0$, we obtain that $|\phi(s)| \le M$, namely, $\phi(s) = O(|\tau|^0)$. Next assume $b \ne 0$. We define a holomorpic function:

$$\psi(s) = (s - v_1 + 1)^b = e^{b \log(s - v_1 + 1)},$$

where log takes the principal value. Since

$$\Re(\log(s - v_1 + 1) = \log(|s - v_1 + 1|),$$

we have uniformly on F

$$|\psi(s)| = |s - v_1 + 1|^b \sim |\tau|^b \ (|\tau| \to \infty)$$

Put $\phi_1(s) = \phi(s)/\psi(s)$. Then $\phi_1(s)$ satisfies the same condition as ϕ with $b = 0$, so that by the above result, $\phi_1(s)$ is bounded on F, thus we get the result for arbitrary b.

□

For a holomorphic function $f(z) = \sum_{n=0}^\infty a_n e^{2\pi i n z}$ on $\mathbb{H}$ satisfying condition 4.1, we put $L(s, f) = \sum_{n=1}^\infty a_n n^{-s}$, since $a_n = O(n^v)$, $L(s, f)$ converges absolutely and uniformly on any compact subset of $\Re(s) > 1 + v$, so that it is holomorphic on $\Re(s) > 1 + v$. We call $L(s, f)$ the *Dirichlet series associated with f.* For $N > 0$, we put $\Lambda_N(s; f) = (2\pi/\sqrt{N})^{-s}\Gamma(s)L(s, f)$

Theorem 4.4. *Let $f(z) = \sum_{n=0}^\infty a_n e^{2\pi i n z}$ and $g(z) = \sum_{n=0}^\infty b_n e^{2\pi i n z}$ be holomorphic funtions satisfying the condition 4.1. For positive k and N, the following condition (1) and (2) are equivalent.*

(1) $g(z) = (-i\sqrt{N}z)^{-k} f(-1/Nz)$

(2) Both $\Lambda_N(s; f)$ and $\Lambda_N(s; g)$ can be analytically continued to the whole s-plane, satisfy the functional equation

$$\Lambda_N(s; f) = \Lambda_N(k - s; g),$$

and

$$\Lambda_N(s; f) + \frac{a_0}{s} + \frac{b_0}{k - s}$$

is holomorphic on the whole s-plane and bounded on any vertical strip $\sigma_1 \leq \Re(s) \leq \sigma_2$.

Proof. From (1)$\Rightarrow$ (2): similar to the proof of proposition 3.5.

From (2)$\Rightarrow$(1) By the Mellin inversion formular: For $\sigma > 0$,

$$e^{-t} = \frac{1}{2\pi i}\int_{\Re(s)=\sigma}\Gamma(s)t^{-s}ds,$$

let $t = 2\pi ny$, we have

$$f(iy) = \frac{1}{2\pi i}\sum_{n=1}^{\infty} a_n \int_{\Re(s)=\alpha}(2\pi ny)^{-s}\Gamma(s)ds + a_0$$

for any $\alpha > 0$. If $\alpha > v+1$, then $L(s,f) = \sum_{n=1}^{\infty} a_n n^{-s}$ is uniformly convergent and bounded on $Re(s) = \alpha$, so that by the Stirling's estimate

$$\Gamma(s) \sim \sqrt{2\pi}\tau^{\sigma-1/2}e^{-\pi|\tau|/2},\ (s = \sigma + i\tau, |\tau| \to \infty),$$

$\Lambda_N(s;f) = (2\pi/\sqrt{N})^{-s}\Gamma(s)L(s;f)$ is absolutely integrable. Therefore we can exchange the order of summation and integration, and

$$f(iy) = \frac{1}{2\pi i}\int_{\Re(s)=\alpha}(\sqrt{N}y)^{-s}\Lambda_N(s;f)ds + a_0.$$

Since $L(s;f)$ is bounded on $\Re(s) = \alpha$, we see, for any $\mu > 0$,

$$|\Lambda_N(s;f)| = O(|\mathrm{Im}(s)|^{-\mu})\ (|\mathrm{Im}(s)| \to \infty)$$

on $\Re(s) = \alpha$ by Stirling's estimate. Next take β so that $k - \beta > v+1$. A similar argument implies that for any $\mu > 0$,

$$|\Lambda_N(s;f)| = |\Lambda_N(k-s;g)| = O(|\mathrm{Im}(s)|^{-\mu})\ (|\mathrm{Im}(s)| \to \infty)$$

on $\Re(s) = \beta$. By assumtion,

$$\Lambda_N(s;f) + \frac{a_0}{s} + \frac{b_0}{k-s}$$

is bounded on the domain $\beta \leq \Re(s) \leq \alpha$. Hence for any $\mu > 0$, we see by 4.3

$$|\Lambda_N(s;f)| = O(|\mathrm{Im}(s)|^{-\mu})\ (|\mathrm{Im}(s)| \to \infty)$$

holds unifomly on the domain $\beta \leq \Re(s) \leq \alpha$. Furthermore we assume that $\alpha > k$ and $\beta < 0$. Since $(\sqrt{N}y)^{-s}\Lambda_N(s;f)$ has simple poles at $s = 0$ and $s = k$ with residues $-a_0$ and $(\sqrt{N}y)^{-k}b_0$, respectively, we can change the integral paths from $\Re(s) = \alpha$ to $\Re(s) = \beta$ and obtain

$$f(iy) = \frac{1}{2\pi i}\int_{\Re(s)=\beta}(\sqrt{N}y)^{-s}\Lambda_N(s;f)ds + (\sqrt{N}y)^{-k}b_0.$$

By the function equation,

$$\begin{aligned} f(iy) &= \frac{1}{2\pi i}\int_{\Re(s)=\beta}(\sqrt{N}y)^{-s}\Lambda_N(k-s;g)ds + (\sqrt{N}y)^{-k}b_0 \\ &= \frac{1}{2\pi i}\int_{\Re(s)=k-\beta}(\sqrt{N}y)^{s-k}\Lambda_N(s;g)ds + (\sqrt{N}y)^{-k}b_0 \\ &= (\sqrt{N}y)^{-k}g(-1/(iNy)) \end{aligned}$$

Since $f(z)$ and $g(z)$ are holomorpic on $\mathbb{H}$, we obtain

$$f(z) = (\sqrt{N}z/i)^{-k}g(-1/Nz),$$
$$g(z) = (-i\sqrt{N}z)^{-k}f(-1/Nz)$$

□

Since $\Gamma(1)$ is generated by two elements T and S, we can easily characterize an element $f(z)$ of $\mathcal{M}_k(\Gamma(1))$ by the functional equation of $L(s,f)$ and obtain an

Corollary 4.5. *Let k be an even integer ≥ 2, assume a holomorphic function $f(z)$ on $\mathbb{H}$ satisfies the condition. Then $f(z)$ belongs to $\mathcal{M}_k(\Gamma(1))$ if and only if $\Lambda(s;f) = (2\pi)^{-s}\Gamma(s)L(s,f)$ can be analytically continued to the whole s-plane,*

$$\Lambda(s;f) + \frac{a_0}{s} + \frac{(-1)^{k/2}a_0}{k-s}$$

is holomorphic on $\mathbb{H}$ and bounded on any vertical strip and satisfies the functional equation :

$$\Lambda(s;f) = (-1)^{k/2}\Lambda(k-s;f).$$

Moreover if $a_0 = 0$, then $f(z)$ is a cusp form.

This part is based on [4] and [3]. I also used some proofs from the book [5].

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