The non-holomorphic Eisenstein series

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The point of this talk will be to present an another type of form, so-called Maass form, and an example of such forms, the non-holomorphic Eisenstein series.

The reason to study some other functions than modular forms is that the condition of holomorphicity is very restrictive - for instance we have seen that there is no nontrivial weight 0 (or *automorphic*) modular forms. We want to somehow relax this condition; for example we could require real analytic functions rather than complex analytic ones. At this point, we ask to ourselves : how can we generate analytic functions ? A way to answer this question is for example to define an elliptic operator; its eigenvectors will be analytic functions. In particular, to get automorphicity, we would like to study an $SL(2,\mathbb{Z})$-invariant operator. This desired operator will be the hyperbolic Laplace operator, which is defined as

$$\Delta := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Then we will define Maass forms, which are automorphic forms which are eigenvectors of the hyperbolic Laplace operator and are of moderate growth.

Historically, Maass forms were defined in 1949 by Hans Maass, to generalise a construction of Hecke's which associated to imaginary quadratic fields a theta function from which one could recover the field's Dedekind zeta function. Maass wanted to find an equivalent for real quadratic fields, and these functions were naturally eigenfunctions of the hyperbolic Laplace operator, of moderate growth, and invariant under the action of the congruence subgroup $\Gamma(D)$.

Secondly, we would like to find an example of Maass form. The most obvious eigenvector of the Laplace operator is simply y^s for $s \in \mathbb{C}$. Now, to obtain an automorphic form, we average it over $SL(2,\mathbb{Z})$, we obtain the following series :

$$E(z,s) = \pi^{-s}\ \Gamma(s)\ \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{y^s}{|mz+n|^{2s}}.$$

This series is the non-holomorphic Eisenstein series.

1 Maass forms

Let's start by defining the hyperbolic Laplace operator.

Definition 1.1. The (weight 0) **hyperbolic Laplace operator** on $\mathcal{H}$ is defined as :

$$\Delta : C^\infty(\mathcal{H}) \to C^\infty(\mathcal{H})$$
$$\Delta := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Definition 1.2. A **singular Maass form** for $SL(2, \mathbb{Z})$ is a smooth function f on $\mathcal{H}$ which satisfies the following two conditions :

1. f is *automorphic*, i.e. $f(\gamma z) = f(z)\ \forall \gamma \in SL(2, \mathbb{Z})$,
2. f is an *eigenvector* of the hyperbolic Laplace operator Δ, i.e. $\Delta f = \lambda f$ for some $\lambda \in \mathbb{C}$.

Definition 1.3. A **weak Maass form** for $SL(2, \mathbb{Z})$ is a singular Maass form f which has the additional property of growing exponentially, i.e. there exists a constant $C > 0$ such that

$$f(z) = \mathcal{O}(e^{Cy}) \text{ as } y = Im(z) \to \infty.$$

Definition 1.4. A **Maass form** for $SL(2, \mathbb{Z})$ is a singular Maass form f which has the additional property of growing polynomially, i.e. there exists a constant $C > 0$ such that

$$f(z) = \mathcal{O}(y^C) \text{ as } y = Im(z) \to \infty.$$

Now observe that the action of $SL(2, \mathbb{Z})$ on $\mathcal{H}$ induces a natural action of $SL(2, \mathbb{Z})$ on $C^\infty(\mathcal{H})$, that is, for all $\gamma \in SL(2, \mathbb{Z})$ there exists a homomorphism θ_γ defined by :

$$\begin{aligned} \theta_\gamma : C^\infty(\mathcal{H}) &\to C^\infty(\mathcal{H}) \\ f \quad &\mapsto \gamma \cdot f, \end{aligned}$$

where $(\gamma \cdot f)(z) = f(\gamma \cdot z)$.

Lemma 1.5. *The hyperbolic Laplace operator Δ on $\mathcal{H}$ is $SL(2, \mathbb{Z})$-invariant.*

The proof of the following lemma is principally technical calculations, where the tricks are to use the Cauchy-Riemann equations and the chain-rule. This lemma will become important in the second part of this talk to show that the non-holomorphic Eisenstein series is automorphic.

Proof. Let $\gamma \in SL(2, \mathbb{Z})$. We have to show that

$$\theta_\gamma \circ \Delta = \Delta \circ \theta_\gamma,$$

where θ_γ has been defined before stating this lemma.

In other words, let $f \in C^\infty(\mathcal{H})$ and $z = x + iy \in \mathbb{C}$, we have to show that

$$\Delta(f)(\gamma(z)) = \Delta(f \circ \gamma)(z).$$

Set $\gamma(z) = \gamma(x, y) = u(x, y) + iv(x, y)$. We will use simpler notations for partial derivatives in this proof to make this calculations more readable, e.g. f_u instead of $\frac{\partial}{\partial u} f$.

On the left hand side of the equation, we have :

$$\Delta(f)(\gamma(z)) = -v^2 \left(f_{uu}(u, v) + f_{vv}(u, v) \right).$$

On the right hand side of the equation, we obtain :

$$\begin{aligned}
\Delta(f \circ \gamma) &= -y^2 \left(\frac{\partial^2}{\partial x^2} f(u, v) + \frac{\partial^2}{\partial y^2} f(u, v) \right) \\
&= -y^2 \left(\frac{\partial}{\partial x} \left(f_u(u, v) u_x + f_v(u, v) v_x \right) \frac{\partial}{\partial y} \left(f_u(u, v) u_y + f_v(u, v) v_y \right) \right) \\
&= -y^2 \Big((f_{uu} u_x + f_{vu} v_x) u_x + f_u u_{xx} + (f_{vu} u_x + f_{vv} v_x) v_x + f_v v_{xx} \\
&\quad + (f_{uu} u_y + f_{vu} v_y) u_y + f_u u_{yy} + (f_{vu} u_y + f_{vv} v_y) v_y + f_v v_{yy} \Big).
\end{aligned}$$

Now, observe that $\gamma(x, y)$ is holomorphic, thus it satisfies the Cauchy-Riemann equations, that is :

$$u_x = v_y \text{ and } u_y = -v_x$$

and this implies also that $u_{xx} + u_{yy} = 0$. Using these equations in the previous calculations, we obtain :

$$\begin{aligned}
\Delta(f \circ \gamma) &= -y^2 (u_x^2 + v_x^2)(f_{uu}(u, v) + f_{vv}(u, v)) \\
&= -y^2 \left| \frac{d}{dz} \gamma(z) \right|^2 (f_{uu}(u, v) + f_{vv}(u, v))
\end{aligned}$$

We are reduced to showing that

$$-y^2 \left| \frac{d}{dz} \gamma(z) \right|^2 = -v^2.$$

Suppose that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = 1$. Observe that, as it has been shown in talk 1,

$$v(x, y) = \frac{y}{|cz + d|^2}$$

Then

$$\begin{aligned}
-y^2\left|\frac{d}{dz}\gamma(z)\right|^2 &= -y^2|v_y + iv_x|^2 \\
&= -y^2\left(\left(\frac{|cz+d|^2 - 2c^2y^2}{|cz+d|^4}\right)^2 + \left(\frac{2cy(cx+d)}{|cz+d|^4}\right)^2\right) \\
&= -\frac{y^2}{|cz+d|^4} \\
&= -v^2.
\end{aligned}$$

□

2 The non-holomorphic Eisenstein series and its first properties

We first define the non-holomorphic Eisenstein series for $SL(2,\mathbb{C})$. The aim of this section will be to show that this series is a singular Maass form. In fact, it is a Maass form; the polynomial growth condition will be shown in the third section.

Definition 2.1. Let $z = x + iy \in \mathcal{H}$ and $s \in \mathbb{C}$. The **non-holomorphic Eisenstein series** for $SL(2,\mathbb{C})$ is defined as

$$E(z,s) = \pi^{-s}\ \Gamma(s)\ \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{y^s}{|mz+n|^{2s}}$$

where Γ is the Gamma function which is defined by

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt.$$

This function, unlike the Eisenstein series G_k defined in previous talks for integer k, is not holomorphic, since the complex modulus function isn't.

Lemma 2.2. *The series $E(z,s)$ converges absolutely if $Re(s) > 1$.*

Proof. The convergence boils down to the convergence of the series

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{|mz+n|^{2s}}$$

since the rest is constant in z. One can simply apply the proof given by Aurel and Etienne for absolute convergence on compact sets of their Eisenstein series. □

Theorem 2.3. *The non-holomorphic Eisenstein series $E(z,s)$ is automorphic, that is*
$$E(\gamma(z),s) = E(z,s) \quad \forall \gamma \in SL(2,\mathbb{Z}).$$

To prove this property of the non-holomorphic Eisenstein series, we will first show that it can be written in a different form. First, recall the following particular congruence subgroups :

$$\Gamma(1) = SL(2,\mathbb{Z}) \text{ and } \Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbb{Z} \right\}.$$

and the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Lemma 2.4. *We have a bijection of sets :*

$$\begin{aligned} \left\{(c,d) \in \mathbb{Z}^2 \textit{ coprimes}\right\} &\leftrightarrow \{\gamma \in \Gamma_\infty \setminus \Gamma(1)\} \\ (c,d) &\leftrightarrow \gamma \textit{ with } (c,d) \textit{ as bottom row} \end{aligned}$$

Proof. The fact that the bottom row of an element of $SL(2,\mathbb{Z})$ has to be formed of coprime integers comes from the Bezout identity.

The point is to show that two elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ are equivalent in $\Gamma_\infty \setminus \Gamma(1)$ if and only if they have the same bottom row.

$$\begin{aligned} &\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \text{ are equivalent in } \Gamma_\infty \setminus \Gamma(1) \\ &\Leftrightarrow \exists \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty \text{ such that } \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ &\Leftrightarrow \exists n \in \mathbb{Z} \text{ such that } \begin{cases} a + nc = e \\ b + nd = f \\ c = g \\ h = d \end{cases}. \end{aligned}$$

Thus, if two elements in $\Gamma_\infty \setminus \Gamma(1)$, they have the same bottom row. On the other hand, if two elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\begin{pmatrix} e & f \\ c & d \end{pmatrix}$ in $\Gamma_\infty \setminus \Gamma(1)$ have the same bottom row, we are reduced to showing that there exists some $n \in \mathbb{Z}$ such that $\begin{cases} a + nc = e \\ b + nd = f \end{cases}$. We suppose that d and c are non zero, since the other case uses the same kind of arguments. Using the fact that the determinant of matrices in $\Gamma(1)$ is 1, we have the following equalities :

$$\begin{cases} ad - bc = 1 \\ ed - fc = 1 \end{cases} \Rightarrow (e-a)d = (f-b)c$$

By the Bezout identity, d and c are coprime, thus d divides $f-b$ and c divides $e-a$, and we choose $n := (f-b)/d = (e-a)/c \in \mathbb{Z}$. $\square$

Lemma 2.5. $E(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\setminus\Gamma(1)} Im(\gamma(z))^s.$

Proof. Let's begin this proof by showing the following identity

$$\frac{Im(z)^s}{|mz+n|^{2s}} = N^{-2s} Im(\gamma(z))^s \tag{1}$$

with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and (c,d) a pair of coprime integers such that there exists some positive integer N which satisfies $(m,n) = (Nc, Nd)$.

$$\begin{aligned}
\mathrm{Im}(\gamma(z))^s &= \frac{\mathrm{Im}(z)^s}{|cz+d|^{2s}} \\
&= \frac{\mathrm{Im}(z)^s}{|N^{-1}mz + N^{-1}n|^{2s}} \\
&= N^{2s}\frac{\mathrm{Im}(z)^s}{|mz+n|^{2s}}.
\end{aligned}$$

Now, observe that the following two sets are equal :

$$\big\{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}\big\} = \left\{(m,n)\in\mathbb{Z}^2 \;\middle|\; \begin{array}{c}\exists c,d \text{ coprime in } \mathbb{Z}, \exists N\in\mathbb{N} \\ \text{such that } (m,n)=(Nc,Nd)\end{array}\right\}.$$

In other words, we have the identity :

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} = \sum_{N\in\mathbb{N}} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \text{coprime}}}.$$

Now, consider the group $\Gamma_\infty \setminus \Gamma(1)$. Using the previous lemma, we understand that summing over all pairs of coprime integers is the same as summing over the elements of the right coset $\Gamma_\infty \setminus \Gamma(1)$.

Thus we have, by using (1) :

$$\begin{aligned}
E(z,s) &= \pi^{-s}\ \Gamma(s)\ \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{y^s}{|mz+n|^{2s}} \\
&= \pi^{-s}\ \Gamma(s)\ \frac{1}{2}\sum_{N\in\mathbb{N}}\sum_{\gamma\in\Gamma_\infty\setminus\Gamma(1)} N^{-2s}\mathrm{Im}(\gamma(z))^s \\
&= \pi^{-s}\ \Gamma(s)\ \frac{1}{2}\left(\sum_{N\in\mathbb{N}} N^{-2s}\right)\left(\sum_{\gamma\in\Gamma_\infty\setminus\Gamma(1)} \mathrm{Im}(\gamma(z))^s\right) \\
&= \pi^{-s}\ \Gamma(s)\ \zeta(2s)\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\setminus\Gamma(1)} \mathrm{Im}(\gamma(z))^s
\end{aligned}$$

□

Now we are ready to prove the automorphicity of the non-holomorphic Eisenstein series :

Proof of theorem 2.3. Using the previous lemma, we have :

$$\begin{aligned}
E(\tau(z),s) &= \pi^{-s}\Gamma(s)\zeta(2s)\frac{1}{2}\sum_{\gamma\in\Gamma_\infty\setminus\Gamma(1)} \mathrm{Im}(\gamma\tau(z))^s \\
&= \pi^{-s}\Gamma(s)\zeta(2s)\frac{1}{2}\sum_{\sigma\in\Gamma_\infty\setminus\Gamma(1)} \mathrm{Im}(\sigma(z))^s \\
&= E(z,s).
\end{aligned}$$

We have applied a change of variables $\sigma = \gamma\tau$ at the second line. □

Theorem 2.6. *The non-holomorphic Eisenstein series is an eigenvector of the hyperbolic Laplace operator Δ with eigenvalue $s(1-s)$.*

Proof. First, let's prove that for all $z = x + iy \in \mathcal{H}$,

$$\Delta(\mathrm{Im}(z))^s = s(1-s)\mathrm{Im}(z)^s.$$

This is a direct calculation :

$$\begin{aligned}
\Delta(\mathrm{Im}(z))^s &= -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) y^s \\
&= -y^s\left(\frac{\partial^2}{\partial y^2}(y^s)\right) \\
&= (-y^s)s(s-1)y^{s-2} \\
&= s(1-s)\mathrm{Im}(z)^s.
\end{aligned}$$

Now recall by lemma 1.5 that the hyperbolic Laplace operator is $SL(2, \mathbb{Z})$-invariant, then

$$\Delta(\mathrm{Im}(\gamma(z)))^s = \theta_\gamma \Delta(\mathrm{Im}(z))^s,$$

where θ_γ was defined in the first section by

$$\begin{aligned} \theta_\gamma : C^\infty(\mathcal{H}) &\to C^\infty(\mathcal{H}) \\ f \quad &\mapsto \ \gamma \cdot f, \end{aligned}$$

where $(\gamma \cdot f)(z) = f(\gamma \cdot z)$.

Using this two identities, we can prove that the Eisenstein series is an eigenvector of Δ, using the appropriate form of $E(z, s)$ that we have found in lemma 2.4 and the fact that Δ is linear:

$$\begin{aligned} \Delta E(z, s) &= \pi^{-s}\Gamma(s)\zeta(2s)\frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} \Delta(\mathrm{Im}(\gamma(z)))^s \\ &= \pi^{-s}\Gamma(s)\zeta(2s)\frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} \theta_\gamma \Delta(\mathrm{Im}(z))^s \\ &= \pi^{-2}\Gamma(s)\zeta(2s)\frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} s(1-s)\theta_\gamma(\mathrm{Im}(z))^s \\ &= s(1-s)\pi^{-s}\Gamma(s)\zeta(2s)\frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma(1)} \mathrm{Im}(\gamma(z))^s \\ &= s(1-s)E(z, s). \end{aligned}$$

□

Proposition 2.7. *The non-holomorphic Eisenstein series $E(z, s)$ is a singular Maass form.*

Proof. This follows directly from theorem 2.3 and theorem 2.5. □

3 The meromorphic continuation

The aim of this section will be to prove that the non-holomorphic Eisenstein series can be meromorphically continued to all of $\mathbb{C}$, with simple poles at $s = 0$ and $s = 1$.

To this aim, we will calculate the Fourier expansion for the series, and observe that all terms in the expansion can be adequately continued; we then conclude with the identity theorem.

This will also give us a functional equation for the non-holomorphic Eisenstein series, namely $E(z, s) = E(z, 1 - s)$.

3.1 The K-Bessel function

This section introduces an auxiliary function which we will then use in calculating of the Fourier expansion for the non-holomorphic Eisenstein series.

Definition 3.1. The *K-Bessel function* $K_s(y)$ is defined, for $y > 0$, $s \in \mathbb{C}$, by

$$K_s(y) = \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2} t^s \frac{dt}{t}.$$

The function is also known as the *Macdonald Bessel function.*

One can already note that the integrand decays rapidly as $t \to 0$ or $t \to \infty$ since $y > 0$, so the function is well-defined for all values of s.

Lemma 3.2. *The K-Bessel function is even in s, that is*

$$K_s(y) = K_{-s}(y).$$

Proof. The measure $\frac{dt}{t}$ is invariant under $t \mapsto t^{-1}$. Hence,

$$K_s(y) = \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2} t^s \frac{dt}{t} = \frac{1}{2}\int_0^\infty e^{-y(t^{-1}+t)/2} t^{-s} \frac{dt}{t} = K_{-s}(y).$$

□

Lemma 3.3. *If $y > 4$, we have*

$$|K_s(y)| \leq e^{-y/2} K_{Re(s)}(2). \tag{2}$$

Proof. If a, b are both greater than 2, then $ab > a + b$; hence $e^{-ab} < e^{-(a+b)} = e^{-a}e^{-b}$. Apply this with $a = y/2$, $b = t + t^{-1}$ and get

$$e^{-y(t+t^{-1})/2} \leq e^{-y/2} e^{-(t+t^{-1})}. \tag{3}$$

Multiplying (3) by $t^{Re(s)}$ and integrating with respect to t, one gets

$$\begin{aligned} |K_s(y)| &= \left| \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2} t^s \frac{dt}{t} \right| \leq \frac{1}{2}\int_0^\infty \left| e^{-y(t+t^{-1})/2} t^s \right| \frac{dt}{t} \\ &\leq \frac{1}{2}\int_0^\infty \left| e^{-y/2} e^{-(t+t^{-1})} \right| t^{\mathrm{Re}(s)} \frac{dt}{t} = e^{-y/2} K_{Re(s)}(2). \end{aligned}$$

□

Lemma 3.4. *When $s = 1/2$, we have*

$$K_{\frac{1}{2}}(y) = \sqrt{\frac{\pi}{2y}} e^{-y}. \tag{4}$$

Proof. Indeed we have

$$\begin{aligned} K_{\frac{1}{2}}(y) &= \frac{1}{2}\int_0^\infty e^{-y(t+t^{-1})/2} t^{1/2}\frac{dt}{t} \\ &= \frac{1}{2\sqrt{y}}\int_0^\infty e^{-(t+y/t)/2} t^{1/2}\frac{dt}{t}. \end{aligned}$$

Here we have used the change of variables $t \mapsto \frac{t}{y}$ and the fact that the measure $\frac{dt}{t}$ is invariant under multiplication (we will use such arguments again in the course of the following proofs). We can then define

$$h(y) := \sqrt{y}K_{\frac{1}{2}}(y) = \frac{1}{2}\int_0^\infty e^{-(t+y/t)/2} t^{1/2}\frac{dt}{t}.$$

Differentiating under the integral gives us

$$\begin{aligned} h'(y) &= -\frac{y}{2}\int_0^\infty e^{-(t+y^2/t)/2} t^{-1/2}\frac{dt}{t} \\ &= -\frac{y}{2}\int_0^\infty e^{-(t^{-1}+y^2 t)/2} t^{1/2}\frac{dt}{t} \\ &= -\frac{1}{2}\int_0^\infty e^{-(t+y/t)/2} t^{1/2}\frac{dt}{t} = -h(y). \end{aligned}$$

The second equality comes from $\frac{dt}{t}$ being invariant under $t \mapsto t^{-1}$, the second from the change of variables $t \mapsto t/y^2$. Hence

$$h'(y) = -h(y) \Rightarrow h(y) = Ce^{-y}$$

and we can find C by calculating $h(0)$.

$$C = \frac{1}{2}\int_0^\infty e^{-t/2} t^{1/2}\frac{dt}{t} = \frac{1}{\sqrt{2}}\int_0^\infty e^{-t} t^{1/2}\frac{dt}{t} = \frac{1}{\sqrt{2}}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{2}},$$

using the substitution $t \mapsto 2t$. Hence

$$h(y) = \sqrt{\frac{\pi}{2}}e^{-y} \Rightarrow K_{\frac{1}{2}}(y) = \sqrt{\frac{\pi}{2y}}e^{-y}.$$

□

Proposition 3.5. *If* $\mathrm{Re}(s) > 1/2$ *and* r *is real, then*

$$\left(\frac{y}{\pi}\right)^s \Gamma(s)\int_{-\infty}^\infty (x^2+y^2)^{-s} e^{2\pi i r x} dx = \begin{cases} \pi^{-s+1/2}\Gamma(s-\frac{1}{2})y^{1-s} & \text{if } r = 0 \\ 2|r|^{s-1/2}\sqrt{y}K_{s-1/2}(2\pi|r|y) & \text{if } r \neq 0 \end{cases}. \quad (5)$$

Proof. Recall the definition of the Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt.$$

We can thus write the left side of (5) as

$$\int_{-\infty}^\infty \int_0^\infty e^{-t}\left(\frac{ty}{\pi(x^2+y^2)}\right)^s e^{2\pi irx}\frac{dt}{t}dx.$$

Since $\mathrm{Re}(s) > \frac{1}{2}$, the integral converges absolutely, so we can use Fubini's theorem. Secondly, using the change of variables $t \mapsto \frac{ty}{\pi(x^2+y^2)}$ (recall the measure dt/t is multiplicatively invariant), we then change the above to

$$\int_0^\infty \int_{-\infty}^\infty e^{-\pi t(x^2+y^2)/y}t^s e^{2\pi irx}dx\frac{dt}{t}. \tag{6}$$

We then use that the Fourier transform of a Gaussian function $f(x) = e^{-ax^2}$ is $\hat{f}(y) = \sqrt{\frac{\pi}{a}}e^{-\pi^2y^2/a}$ to say

$$\int_{-\infty}^\infty e^{-t\pi x^2/y}e^{2\pi irx}dx = \begin{cases}\sqrt{\frac{y}{t}} & \text{if } r=0\\ \sqrt{\frac{y}{t}}e^{-y\pi r^2/t} & \text{if } r\neq 0\end{cases}$$

(we are evaluating the Fourier transform at r). Plugging this back into (6), we get

$$\begin{aligned}\int_0^\infty \int_{-\infty}^\infty e^{-\pi t(x^2+y^2)/y}t^s e^{2\pi irx}dx\frac{dt}{t} &= \int_0^\infty \sqrt{\frac{y}{t}}t^s e^{-\pi ty}\frac{dt}{t}\\ &= \sqrt{y}\frac{1}{\pi y}\int_0^\infty \sqrt{\frac{\pi y}{t}}\left(\frac{t}{\pi y}\right)^{s-1}e^{-t}dt\\ &= \pi^{-s}y^{-s}\pi^{1/2}y\int_0^\infty t^{s-3/2}e^{-t}dt\\ &= \pi^{-s+1/2}y^{1-s}\Gamma(s-1/2)\end{aligned}$$

when $r=0$, and

$$\begin{aligned}\int_0^\infty \int_{-\infty}^\infty e^{-\pi t(x^2+y^2)/y}t^s e^{2\pi irx}dx\frac{dt}{t} &= \int_0^\infty \sqrt{\frac{y}{t}}e^{-y\pi r^2/t}t^s e^{-\pi ty}\frac{dt}{t}\\ &= \sqrt{y}\int_0^\infty t^{s-1/2}|r|^{s-1/2}e^{-\pi y(|r|/t+|r|t)}\frac{dt}{t}\\ &= |r|^{s-1/2}\sqrt{y}\int_0^\infty t^{s-1/2}e^{-\pi y|r|(t+t^{-1})}\frac{dt}{t}\\ &= 2|r|^{s-1/2}\sqrt{y}K_{s-1/2}(2\pi|r|y)\end{aligned}$$

when $r\neq 0$, using the change of variables $t\mapsto \frac{t}{|r|}$. □

3.2 The Fourier expansion of the series

Since the non-holomorphic Eisenstein series is fully automorphic, in particular with $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we have

$$E(z+1, s) = E(Tz, s) = E(z, s).$$

Hence we have a Fourier expansion

$$E(z, s) = \sum_{r=-\infty}^{\infty} a_r(y, s) e^{2\pi i r x}$$

with

$$a_r(y, s) = \int_0^1 E(x + iy, s) e^{-2\pi i r x} dx. \tag{7}$$

Our task now is to calculate these Fourier coefficients, and we will be able to get the meromorphic continuation.

Proposition 3.6. *The Fourier coefficients of the non-holomorphic Eisenstein series $E(z, s)$ are*

$$a_0 = \pi^{-s}\,\Gamma(s)\,\zeta(2s)\,y^s + \pi^{s-1}\zeta(2-2s)\Gamma(1-s)y^{1-s} \tag{8}$$

$$a_r = 2|r|^{s-1/2}\sigma_{1-2s}(|r|)\sqrt{y}K_{s-1/2}(2\pi|r|y), \tag{9}$$

where

$$\sigma_{1-2s}(r) = \sum_{m|r} m^{1-2s}.$$

Proof. Expanding out the definition (7) of the coefficients, we get

$$a_r(y, s) = \int_0^1 \pi^{-s}\,\Gamma(s)\,\frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{y^s}{|mx + miy + n|^{2s}} e^{-2\pi i r x} dx.$$

Since both sum and integral converge absolutely, we can manipulate them. First we look at those terms in the sum with $m = 0$. Since in this case the above integral does not depend on x, these terms can only contribute to the Fourier coefficient a_0; call this contribution a_0'' Indeed the integral becomes

$$a_0'' = \frac{1}{2}\int_0^1 \pi^{-s}\,\Gamma(s) \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{y^s}{|n|^{2s}} dx = \pi^{-s}\,\Gamma(s)\,\zeta(2s)\,y^s.$$

Now we consider the terms with $m \neq 0$. Since $m \neq 0$ implies $(m, n) \neq (0, 0)$, we need no restrictions on n. Their contribution (which we may write $a_r'(y, s)$, and note $a_r = a'_r$ whenever $r \neq 0$, $a_0 = a'_0 + a''_0 = a'_0 + \pi^{-s}\,\Gamma(s)\,\zeta(2s)\,y^s)$ is thus:

$$
\begin{aligned}
a'_{-r}(y,s) &= \int_0^1 \pi^{-s}\ \Gamma(s)\ \frac{1}{2} \sum_{m\in\mathbb{Z}\setminus\{0\}} \sum_{n\in\mathbb{Z}} \frac{y^s}{|mx+miy+n|^{2s}} e^{2\pi i r x} dx \\
&= \pi^{-s}\ \Gamma(s) \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \int_0^1 \frac{y^s}{|mx+miy+n|^{2s}} e^{2\pi i r x} dx \\
&= \pi^{-s}\ \Gamma(s)\ y^s \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \int_0^1 \frac{1}{((mx+n)^2+(my)^2)^s} e^{2\pi i r x} dx \\
&= \pi^{-s}\ \Gamma(s)\ y^s \sum_{m=1}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{n=km}^{(k+1)m} \int_0^1 \frac{1}{((mx+n)^2+(my)^2)^s} e^{2\pi i r x} dx \\
&= \pi^{-s}\ \Gamma(s)\ y^s \sum_{m=1}^{\infty} \sum_{n \bmod m} \int_{-\infty}^{\infty} \frac{1}{((mx+n)^2+(my)^2)^s} e^{2\pi i r x} dx.
\end{aligned}
$$

The second equality comes from the fact (m,n) and $(-m,-n)$ contribute equally to the sum, the third from the definition of a complex modulus, while the fourth and fifth are simply set equalities and the fact that the integrand is periodic. Now we use the linear substitution $x \mapsto x + n/m$, and since

$$
\left(x + \frac{n}{m}\right)^2 = x^2 + \frac{2nx}{m} + \frac{n^2}{m^2} = \frac{(mx+n)^2}{m^2},
$$

we have transformed our integral to

$$
\begin{aligned}
&\pi^{-s}\ \Gamma(s)\ y^s \sum_{m=1}^{\infty} \sum_{n \bmod m} e^{2\pi i r n/m} \int_{-\infty}^{\infty} \frac{1}{(m^2x^2+m^2y^2)^s} e^{2\pi i r x} dx \\
&= \pi^{-s}\ \Gamma(s)\ y^s \sum_{m=1}^{\infty} m^{-2s} \sum_{n \bmod m} e^{2\pi i r n/m} \int_{-\infty}^{\infty} \frac{1}{(x^2+y^2)^s} e^{2\pi i r x} dx.
\end{aligned}
$$

Now consider

$$
\sum_{n \bmod m} e^{2\pi i r n/m} = \begin{cases} m & \text{if } m|r \\ 0 & \text{otherwise} \end{cases}
$$

and we get our result

$$
a'_{-r}(y,s) = \pi^{-s}\ \Gamma(s)\ y^s \sum_{m|r} m^{1-2s} \int_{-\infty}^{\infty} \frac{1}{(x^2+y^2)^s} e^{2\pi i r x} dx. \tag{10}
$$

This is where we use the techniques from the K-Bessel function. If $r = 0$, then the condition $m|r$ is vacuous, and so the above becomes

$$\pi^{-s}\,\Gamma(s)\,y^s\zeta(2s-1)\int_{-\infty}^{\infty}\frac{1}{(x^2+y^2)^s}e^{-2\pi irx}dx \underset{(5)}{=} \zeta(2s-1)\pi^{-s+1/2}\Gamma\left(s-\tfrac{1}{2}\right)y^{1-s}$$

whence

$$a_0 = \pi^{-s}\,\Gamma(s)\,\zeta(2s)\,y^s + \zeta(2s-1)\pi^{-s+1/2}\Gamma\left(s-\tfrac{1}{2}\right)y^{1-s}.$$

To get to the form given in the proposition, recall the functional equation

$$\pi^{-w/2}\Gamma\left(\frac{w}{2}\right)\zeta(w) = \pi^{(-1+w)/2}\Gamma\left(\frac{1-w}{2}\right)\zeta(1-w),$$

and apply it to $w = 2s-1$. If $r \neq 0$, we get

$$a_{-r} \underset{(5)}{=} 2|r|^{s-1/2}\sigma_{1-2s}(|r|)\sqrt{y}K_{s-1/2}(2\pi|r|y) = a_r.$$

□

Corollary 3.7. *$E(z,s)$ satisfies a polynomial growth property, namely*

$$E(x+iy,s) = O(y^\sigma)$$

as $y \to \infty$, where $\sigma = \max\{\mathrm{Re}(s), 1-\mathrm{Re}(s)\}$. Hence $E(z,s)$ is a Maass form.

Proof. By lemma 3.3, the non-constant terms rapidly go to 0 when $y \to \infty$, so the function behaves asymptotically like $a_0(y,s)$, and as $y \to \infty$, $a_0 = O(y^s + y^{1-s})$. □

Theorem 3.8. *The non-holomorphic Eisenstein series $E(z,s)$ has meromorphic continuation to all s in $\mathbb{C}$, analytic except for simple poles at $s = 0$ and $s = 1$, where it has a residue of $1/2$ and $-1/2$ respectively. In addition, it satisfies the functional equation*

$$E(z,s) = E(z,1-s).$$

Proof. All the terms a_r, for $r \neq 0$, are products of entire functions and as such are entire ($\sigma_s(x)$ and $K_s(x)$ are entire functions of s, which is obvious from their definitions). As for a_0, $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ has meromorphic continuation to the entire complex plane with poles at 0 and 1.

We can thus use the equation

$$E(z,s) = \sum_{r=-\infty}^{\infty} a_r(y,s)e^{2\pi irx}$$

to define a continuation for $E(z,s)$; the rapid decay property proved in lemma 3.3 shows that this sum converges absolutely.

The functional equation comes from the fact that $a_n(y, s) = a_n(y, 1-s)$. This is obvious for a_0 (the mapping $s \mapsto 1-s$ simply exchanges the two summands). For a_r with $r \neq 0$, it comes from the previously shown identity $K_s = K_{-s}$, hence $K_{s-1/2} = K_{1/2-s} = K_{1-s-1/2}$, and from the fact that in general

$$r^s \sigma_{-2s}(r) = r^s \sum_{m|r} m^{-2s} = r^s \sum_{d_1 d_2 = r} d_1^{-2s} d_2^0 = \sum_{d_1 d_2 = r} d_1^{-s} d_2^s,$$

and similarly $r^{-s}\sigma_{2s}(r) = \sum_{d_1 d_2 = r} d_1^{-s} d_2^s$, applied to $|r|$ and $s - \frac{1}{2}$.

Regarding the poles and the residues, note that a_r is entire whenever $r \neq 0$. As for a_0, $\Gamma(s)$ has a pole at $s = 0$ of residue 1, and $\zeta(s)$ has a pole at $s = 1$ of residue 1. Hence $\pi^{-s}\Gamma(s)\zeta(2s)y^s$ has a pole at $s = 0$ of residue $\frac{1}{2}$ and one at $s = 1/2$ of residue $\frac{1}{2}y^{1/2}$, and $\pi^{s-1}\zeta(2-2s)\Gamma(1-s)y^{1-s}$ has a pole at $s = 1$ of residue $-\frac{1}{2}$ and one at $s = 1/2$ of residue $-\frac{1}{2}y^{1/2}$. Hence the poles at $s = 1/2$ cancel, and the poles of $E(z, s)$ are exactly those given in the theorem. □

The fact that the residues are independent of the value of z can be used for the proof of the analytic class number formula.

Kronecker's limit formula then gives an even more precise expansion of the series at $s = 1$:

Theorem 3.9 (Kronecker limit formula)**.** *The constant term of the Laurent series for $E(z, s)$ at $s = 1$ is*

$$\gamma - log(2) - log(\sqrt{y}|\eta(z)|^2),$$

or in other words

$$E(x + iy, s) = \frac{1}{2(s-1)} + \gamma - log(2) - log(\sqrt{y}|\eta(z)|^2) + O(s-1)$$

as $s \to 1$, where γ is the Euler-Mascheroni constant, and η is the Dedekind êta function

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})$$

or in terms of the discriminant function, $\eta(z)^{24} = \frac{\Delta(z)}{(2\pi)^{12}}$.

Proof. To get this equation, we will need to calculate the Laurent series of $E(z, s)$ at $s = 1$. Once again, we will work first with a_0, then with a_r for $r \neq 0$.

For a_0, we will use a trick and write $\tilde{a}_0 = \frac{\pi^s}{\Gamma(s)} a_0$, explicitly

$$\tilde{a}_0 = \zeta(2s)y^s + \frac{\Gamma(s-1/2)}{\Gamma(s)} \pi^{1/2} \zeta(2s-1) y^{1-s}.$$

Since the $\zeta(2s)y^s$ part is entire at $s = 1$, its contribution is simply

$$\zeta(2)y^1 + O(|s-1|) = \frac{\pi^2 y}{6} + O(|s-1|).$$

For the $G(s) := \frac{\Gamma(s-1/2)}{\Gamma(s)}\pi^{1/2}\zeta(2s-1)y^{1-s}$ part, we use the logarithm to be able to work term by term, recalling the expansions

$$\begin{aligned} \zeta(s) &= \frac{1}{s-1} + \gamma + O(|s-1|) && s \to 1 \\ \log(\Gamma(z)) &= \tfrac{1}{2}\log\pi - (2\log 2 + \gamma)(z - \tfrac{1}{2}) + O(\left|z - \tfrac{1}{2}\right|^2) && \left|z - \tfrac{1}{2}\right| < \tfrac{1}{2} \\ \log(\Gamma(1+z)) &= -\gamma z + O(|z|^2) && |z| < 1. \end{aligned}$$

Hence, using that $\log(1+x) = x + O(|x|^2)$ as $x \to 1$

$$\begin{aligned} \log\zeta(2s-1) &= \log\left(\frac{1}{2s-2} + \gamma + O(|s-1|)\right) \\ &= -\log(2s-2) + \log\left(1 + \gamma(2s-2) + O(|s-1|^2\right) \\ &= -\log(2s-2) + \gamma(2s-2) + O(|s-1|^2) \end{aligned}$$

as $s \to 1$, and

$$\begin{aligned} \log\Gamma(s - \tfrac{1}{2}) &= \tfrac{1}{2}\log\pi - (2\log 2 + \gamma)(s-1) + O(|s-1|^2) && s \to 1 \\ \log\Gamma(s) &= -\gamma(s-1) + O(|s-1|^2) && s \to 1. \end{aligned}$$

These approximations allow us to write $\log G(s)$ when $s \to 1$ as follows:

$$\begin{aligned} \log G(s) &= \tfrac{1}{2}\log\pi - (2\log 2 + \gamma)(s-1) + \gamma(s-1) - \log(2s-2) \\ &\quad + \gamma(2s-2) + \tfrac{1}{2}\log\pi + (1-s)\log y + O(|s-1|^2) \\ &= \log\tfrac{\pi}{2} - \log(s-1) + (2\gamma - 2\log 2 - \log y)(s-1) + O(|s-1|^2). \end{aligned}$$

One can then use the fact that $e^x = 1 + x + O(|x|^2)$ as $x \to 0$ to get

$$\begin{aligned} G(s) &= \tfrac{\pi}{2}\tfrac{1}{s-1}(1 + (2\gamma - 2\log 2 - \log y)(s-1) + O(|s-1|^2)) \\ &= \tfrac{\pi}{2}\tfrac{1}{s-1} + \pi(\gamma - \log 2 - \log\sqrt{y}) + O(|s-1|), \end{aligned}$$

which, along with the fact that

$$\frac{\Gamma(s)}{\pi^s} = \frac{1}{\pi} + O(|s-1|)$$

gives us our final expansion for a_0,

$$a_0 = \tfrac{\pi}{6}y + \frac{1}{2(s-1)} + \gamma - \log 2 - \log\sqrt{y} + O(|s-1|). \tag{11}$$

Now for the other terms,

$$\sum_{r\in\mathbb{Z}\setminus\{0\}} 2|r|^{s-1/2}\sigma_{1-2s}(|r|)\sqrt{y}K_{s-1/2}(2\pi|r|y)e^{2\pi irx}. \tag{12}$$

First of all note that $a_r = a_{-r}$; hence writing $e^{2\pi irx} = \cos(2\pi rx)+i\sin(2\pi rx)$, the sine terms cancel since sin is an odd function, and the cosine terms add up since cos is an even function. So we can rewrite the sum (12) above as

$$4\sum_{r=1}^{\infty} r^{s-1/2}\sigma_{1-2s}(r)\sqrt{y}K_{s-1/2}(2\pi ry)\cos(2\pi rx)$$

We have $r^{s-1/2} = \sqrt{r} + O(|s-1|)$ as $s \to 1$, and

$$\sigma_{1-2s}(r) = \sum_{d|r} d^{1-2s} = \sum_{d|r}(d^{-1} + O(|s-1|)) = \sigma_{-1}(r) + O(|s-1|)$$

as $s \to 1$.

Finally, since (lemma 3.4) $K_{1/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$, we get

$$K_{s-1/2}(2\pi ry) = \frac{1}{2\sqrt{r}\sqrt{y}}e^{-2\pi ry} + O(|s-1|) \qquad s \to 1$$

Hence, as $s \to 1$,

$$\begin{aligned} &4\sum_{r=1}^{\infty} r^{s-1/2}\sigma_{1-2s}(r)\sqrt{y}K_{s-1/2}(2\pi ry)\cos(2\pi rx) \\ =\ &2\sum_{r=1}^{\infty} \sigma_{-1}(r)e^{-2\pi ry}\cos(2\pi rx) + O(|s-1|). \end{aligned}$$

Our goal now is to simplify $S := \sum_{r=1}^{\infty} \sigma_{-1}(r)e^{-2\pi ry}\cos(2\pi rx)$. Writing $q = e^{2\pi iz} = e^{2\pi ix}e^{-2\pi ry}$, we have

$$S = \sum_{r=1}^{\infty} \sigma_{-1}(r)e^{-2\pi ry}\cos(2\pi rx) = \mathrm{Re}\left(\sum_{r=1}^{\infty} \sigma_{-1}(r)(q^r)\right).$$

Now the key step is the following, using the power series for $\log(1+x)$ about $x = 0$:

$$\begin{aligned} \log\prod_{n=1}^{\infty}(1-q^n) &= \sum_{n=1}^{\infty}\log(1-q^n) \\ &= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{q^{nm}}{m} \\ &= -\sum_{N=1}^{\infty}\sum_{d|n}\frac{q^N}{d} \\ &= -\sum_{N=1}^{\infty}\sigma_{-1}(N)q^N, \end{aligned}$$

where the third equality comes from the identification $N := nm$. Hence we can rewrite

$$S = -\operatorname{Re}\left(\log \prod_{r=1}^{\infty} (1 - q^r)\right).$$

Finally, recalling $\operatorname{Re}\log z = \log|z|$, and the definition of η which is $\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty}(1 - q^n)$, we have

$$\log|\eta(z)| = \log\left(e^{-\pi y/12} \prod_{n=1}^{\infty} |1 - q^n|\right) = -\frac{\pi y}{12} - S,$$

or in other words

$$S = -\frac{\pi y}{12} - \log|\eta(z)|.$$

So

$$\sum_{r\in\mathbb{Z}\setminus\{0\}} 2|r|^{s-1/2}\sigma_{1-2s}(|r|)\sqrt{y}K_{s-1/2}(2\pi|r|y)e^{2\pi irx} = -\tfrac{\pi}{6}y - 2\log|\eta(z)| + O(|s-1|), \tag{13}$$

as $s \to 1$.

Putting together (11) and (13), one obtains

$$E(z,s) = \tfrac{\pi}{6}y + \frac{1}{2(s-1)} + \gamma - \log 2 - \log\sqrt{y}) - \tfrac{\pi}{6}y - 2\log|\eta(z)| + O(|s-1|),$$

as $s \to 1$, which, cancelling out the $\frac{\pi}{6}y$ terms and noting $2\log x = \log x^2$, is exactly Kronecker's limit theorem.

□

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